

Conservation Laws III & Stokes Flow

High-resolution schemes, systems, Stokes (Sections 8.5–9.4)

Johannes Herter

MIT CSAIL · Numerical Methods



Lecture II

Conservative finite volume schemes (8.3.10) with monotone fluxes (Lax–Friedrichs, Godunov): correct shock speeds, comparison principle, no new extrema, CFL. But: at most **first order**.

Today.

- 8.5 Higher order by linear reconstruction, made non-oscillatory by **slope limiting**: MUSCL.
- 8.6 Outlook: systems, the Euler equations.
- 9 Finite elements for the **Stokes equations**: saddle points, pressure instability, the inf-sup condition, Taylor–Hood.

8.5 *Higher-order conservative schemes*

Why “high order” for conservation laws?

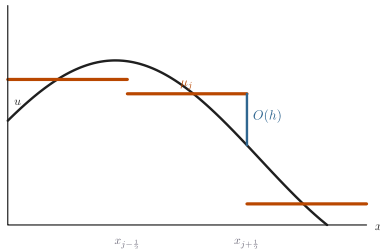
Formally: fluxes consistent of order ≥ 2 (8.4.50). But solutions are not even continuous (§ 8.2.14), so the formal order may not bear on the observed rate. Desired instead:

- for the same spatial resolution, more accurate solutions in global error norms;
- better resolution of local features (shocks, etc.).

In (8.3.10) F sees the cell averages (8.3.4); for $u \in C^1$

$$\frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x_{j+\frac{1}{2}}, t) - u(x, t) dx = O(h): \text{first order only}$$

(unless a lucky cancellation occurs, Ex. 8.4.52).



The cell average misses the value at the cell boundary by $O(h)$.

8.5.1 Piecewise linear reconstruction

Idea: plug better approximations of $u(x_{j\pm\frac{1}{2}}, t)$ into the numerical flux:

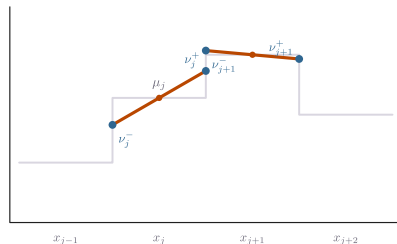
$$\frac{d\mu_j}{dt} = -\frac{1}{h} \left(F(v_j^+, v_{j+1}^-) - F(v_{j-1}^+, v_j^-) \right), \quad (8.5.1)$$

$v_j^\pm$ from a piecewise linear reconstruction from the cell values μ_j :

$$v_j^-(t) := \mu_j(t) - \frac{1}{2}h\sigma_j(t), \quad v_j^+(t) := \mu_j(t) + \frac{1}{2}h\sigma_j(t), \quad (8.5.2)$$

with suitable slopes $\sigma_j(t) = \sigma(\vec{\mu}(t))$.

Analogy: piecewise cubic Hermite interpolation with reconstructed slopes, shape preserving interpolation [33, Sect. 3.4.2]; but now no smoothness is aimed for.



Definition 8.5.3 (Linear reconstruction)

Given an (infinite) mesh $\mathcal{M} := \{]x_{j-1}, x_j[\}_{j \in \mathbb{Z}}$, a linear reconstruction operator $R_{\mathcal{M}}$ is a mapping

$$R_{\mathcal{M}} : \mathbb{R}^{\mathbb{Z}} \rightarrow \{v \in L^{\infty}(\mathbb{R}) : v \text{ linear on }]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[\ \forall j \in \mathbb{Z}\},$$

taking a sequence $\vec{\mu} \in \mathbb{R}^{\mathbb{Z}}$ of cell averages to a possibly discontinuous function $R_{\mathcal{M}}\vec{\mu}$, piecewise linear on dual cells.

Linear reconstruction & (8.5.1): semi-discrete evolution in conservation form, 2-point flux F

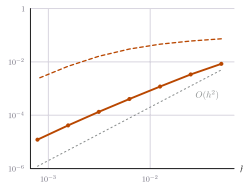
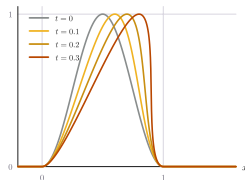
$$\frac{d\mu_j}{dt}(t) = -\frac{1}{h} \left(F(v_j^+(t), v_{j+1}^-(t)) - F(v_{j-1}^+(t), v_j^-(t)) \right), \quad j \in \mathbb{Z}. \quad (8.5.4)$$

Central slope and Example 8.5.10

“Natural” choice, the **central slope** (averaged slope):

$$\sigma_j(t) = \frac{1}{2} \left(\frac{\mu_{j+1}(t) - \mu_j(t)}{h} + \frac{\mu_j(t) - \mu_{j-1}(t)}{h} \right) = \frac{1}{2} \frac{\mu_{j+1}(t) - \mu_{j-1}(t)}{h}. \quad (8.5.9)$$

Taylor: for $u \in C^2$, $|v_j^-(t) - u(x_{j-\frac{1}{2}}, t)|, |v_j^+(t) - u(x_{j+\frac{1}{2}}, t)| = O(h^2)$.

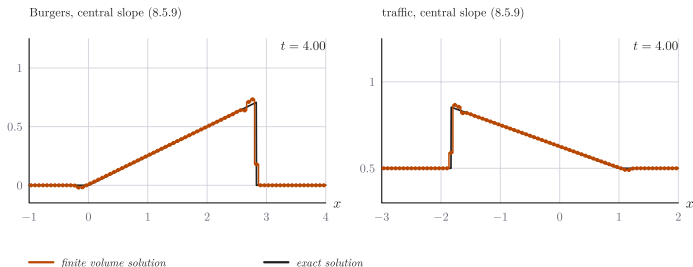


err_1 (solid), err_∞ (dashed), $T = 0.3$

Ex. 8.5.10: Burgers, BUMP, Godunov flux, Heun ($\tau = 0.5h$), $T = 0.3$. **2nd order** in err_1 .

Exs. 8.5.11/8.5.12: central slope

Box data of Exps. 8.3.22, 8.3.23; linear reconstruction with central slope (8.5.9), Godunov flux (8.3.53):

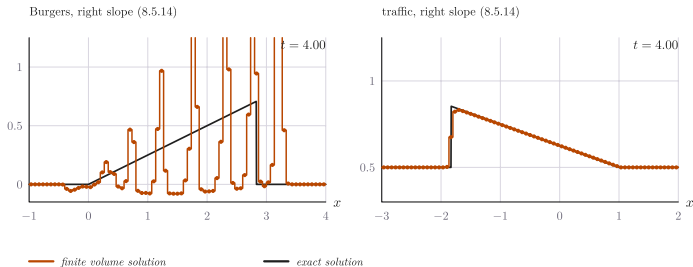


Observation: spurious oscillations near the shock, violating (8.2.42) (cf. Exp. 8.3.22). Is the central slope the culprit?

Exs. 8.5.13/8.5.16: right slope

One-sided slopes for (8.5.2), same setting as Exs. 8.5.11, 8.5.12. Right slope:

$$\sigma_j(t) = \frac{\mu_{j+1}(t) - \mu_j(t)}{h}. \quad (8.5.14)$$



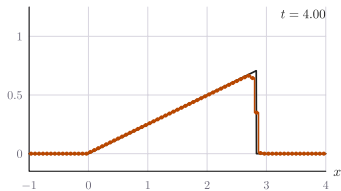
Observation: spurious oscillations / overshoots, massive and global.

Exs. 8.5.13/8.5.16: left slope

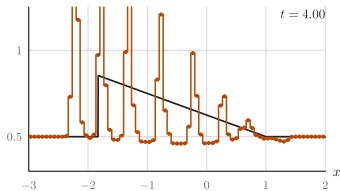
Left slope:

$$\sigma_j(t) = \frac{\mu_j(t) - \mu_{j-1}(t)}{h}. \quad (8.5.15)$$

Burgers, left slope (8.5.15)



traffic, left slope (8.5.15)



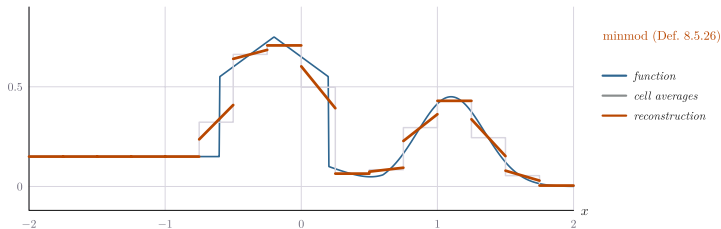
— finite volume solution

— exact solution

Observation: overshoots, moderate and close to the shock.

Ex. 8.5.17: over- and undershoots

Synthetic cell averages of a function with a jump, a kink and a local maximum; slopes (8.5.9), (8.5.14), (8.5.15):



Over- and undershoots *regardless of the slope formula*: linear reconstruction itself triggers oscillations at jumps. (Last frame: minmod, Def. 8.5.26.)

8.5.2 Slope limiting: monotonicity preservation

Definition 8.5.18 (Monotonicity preserving linear interpolation)

A linear reconstruction operator $R_{\mathcal{M}}$ (Def. 8.5.3) is monotonicity preserving, if $(R_{\mathcal{M}}\vec{\mu})(x_j) = \mu_j$ and

$$\mu_j \leq \mu_{j+1} \Rightarrow R_{\mathcal{M}}\vec{\mu} \text{ non-decreasing in }]x_j, x_{j+1}[,$$

$$\mu_j \geq \mu_{j+1} \Rightarrow R_{\mathcal{M}}\vec{\mu} \text{ non-increasing in }]x_j, x_{j+1}[.$$

Constant at plateaus and extrema; cf. shape preserving Hermite interpolation [33, Sect. 3.4.2].

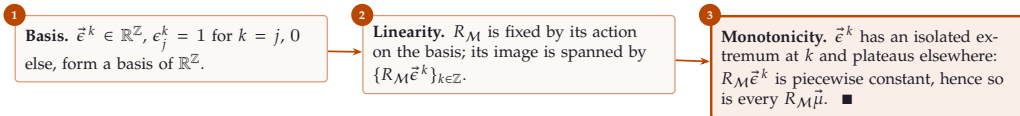
Remark 8.5.19. Range of cell averages respected: $\min_{k \leq l \leq m} \mu_l \leq (R\vec{\mu})(x) \leq \max_{k \leq l \leq m} \mu_l$, $x_k < x < x_m$ (8.5.20), cf. Thm. 8.2.41; no new extrema (8.5.21), cf. Sect. 8.2.7.

Lemma 8.5.24: linear and monotone is trivial

Remark 8.5.22. The reconstructions with (8.5.9), (8.5.14), (8.5.15) are *linear*:
 $R_{\mathcal{M}}(\alpha\vec{\mu} + \beta\vec{v}) = \alpha R_{\mathcal{M}}\vec{\mu} + \beta R_{\mathcal{M}}\vec{v}$ (8.5.23).

Lemma 8.5.24 (Linear monotonicity preserving reconstruction trivial)

Every linear, monotonicity preserving (Def. 8.5.18) linear reconstruction yields piecewise constant functions.



Necessary: a *non-linear* linear reconstruction. Simple consideration: $\mu_{j-1} \leq \mu_j, \mu_j \geq \mu_{j+1} \Rightarrow R_{\mathcal{M}}\vec{\mu} \equiv \text{const on }]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$ (8.5.25): constant at local extrema.

Definition 8.5.26 (Minmod reconstruction)

$(R_{\text{mm}}\vec{\mu})(x) = \mu_j + \sigma_j(x - x_j)$ for $x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}}$, with

$$\sigma_j := \text{minmod}\left(\frac{\mu_{j+1} - \mu_j}{x_{j+1} - x_j}, \frac{\mu_j - \mu_{j-1}}{x_j - x_{j-1}}\right),$$

$$\text{minmod}(v, w) := \begin{cases} v, & vw > 0, |v| < |w|, \\ w, & vw > 0, |w| < |v|, \\ 0, & vw \leq 0. \end{cases}$$

Minmod = a **slope limiter**.

Lemma 8.5.27 (Monotonicity preservation of minmod)

Minmod reconstruction is monotonicity preserving (Def. 8.5.18).

1

W.l.o.g. $\mu_{j+1} \geq \mu_j$. Then $\sigma_j \geq 0$, $\sigma_{j+1} \geq 0$ (minmod is 0 or has the sign of $\mu_{j+1} - \mu_j$), and $\sigma_j, \sigma_{j+1} \leq \frac{\mu_{j+1} - \mu_j}{h}$.



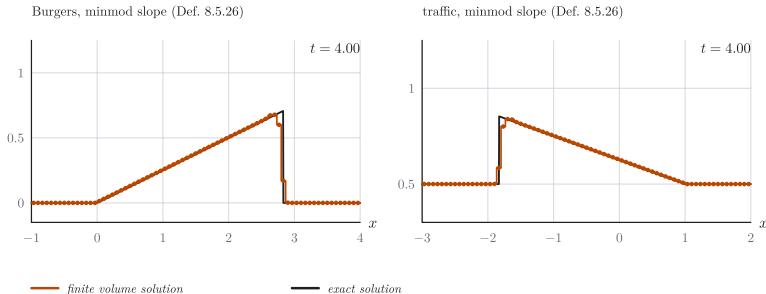
2

Hence on $]x_j, x_{j+1}[$ the reconstruction rises and its jump at $x_{j+\frac{1}{2}}$ is upward:

$$\mu_j + \frac{1}{2}h\sigma_j \leq \frac{1}{2}(\mu_j + \mu_{j+1}) \leq \mu_{j+1} - \frac{1}{2}h\sigma_{j+1}. \blacksquare$$

Exs. 8.5.28/8.5.29: minmod limiter

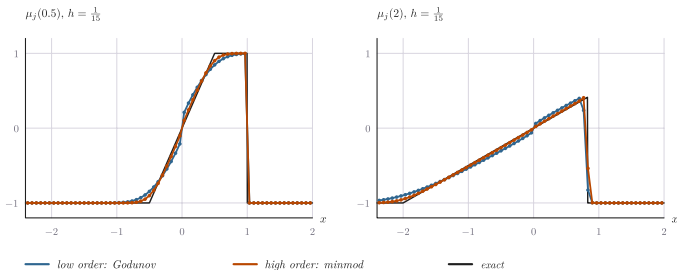
Same settings as Exs. 8.5.11, 8.5.12 with minmod limited slopes $\sigma_j := \minmod\left(\frac{\mu_j - \mu_{j-1}}{h}, \frac{\mu_{j+1} - \mu_j}{h}\right)$,
Godunov flux, $h = \frac{1}{15}$:



Observation: spurious oscillations successfully suppressed!

Ex. 8.5.30: improved resolution

Transonic Burgers data (8.3.43) of Ex. 8.3.42, Godunov flux, with and without minmod-limited reconstruction:



Better resolution of the rarefaction fan, good resolution of the shock: the main rationale for reconstruction.

8.5.3 MUSCL scheme

MUSCL = Monotone Upwind Scheme for Conservation Laws (equidistant mesh, meshwidth h)

- Conservative finite volume discretization (8.5.1) with a monotone consistent 2-point flux, e.g. Godunov (8.3.53);
- piecewise linear reconstruction with minmod slope limiting:

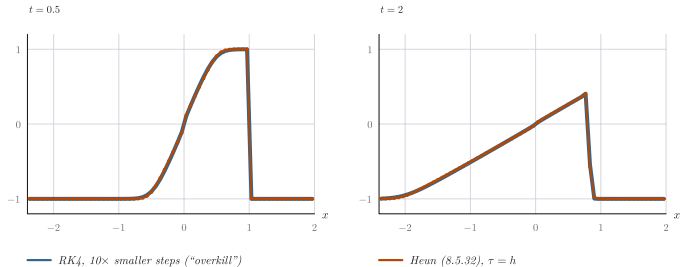
$$v_j^\pm := \mu_j \pm \frac{1}{2} \text{minmod}(\mu_{j+1} - \mu_j, \mu_j - \mu_{j-1}); \quad (8.5.31)$$

- 2nd-order Runge–Kutta timestepping (method of Heun, cf. (8.4.10)) for (8.5.1), with $\mathcal{L}_h(\vec{\mu}) := -\frac{1}{h} (F(v_j^+, v_{j+1}^-) - F(v_{j-1}^+, v_j^-))$:

$$\vec{\kappa} := \vec{\mu}^{(k)} + \tau \mathcal{L}_h(\vec{\mu}^{(k)}), \quad \vec{\mu}^{(k+1)} := \vec{\mu}^{(k)} + \frac{\tau}{2} (\mathcal{L}_h(\vec{\mu}^{(k)}) + \mathcal{L}_h(\vec{\kappa})). \quad (8.5.32)$$

Ex. 8.5.33: 2nd-order timestepping suffices

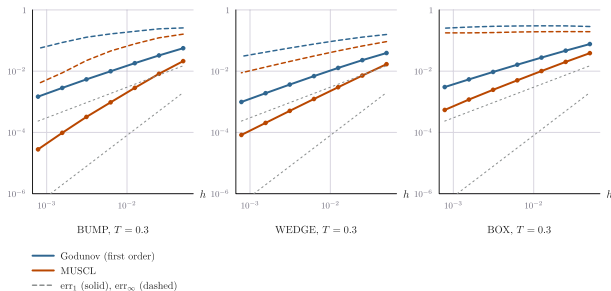
Transonic Burgers data, minmod reconstruction, Godunov flux: Heun with uniform $\tau = h$ against “overkill” integration:



Same accuracy: balance the orders of spatial and temporal discretization, use Heun.

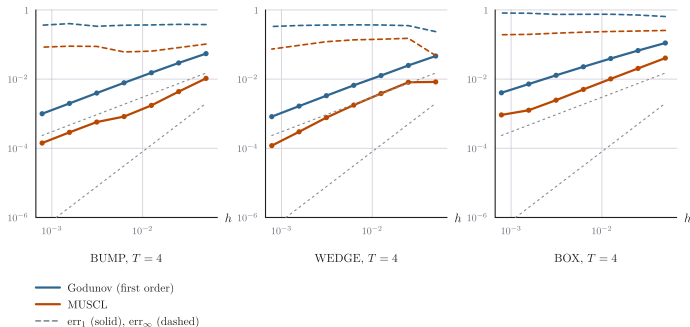
Ex. 8.5.34: convergence of MUSCL

Experiments of Exp. 8.4.47 repeated with MUSCL ($\tau = 0.5h$), against first-order Godunov:



$T = 0.3$: about second order in err_1 for BUMP; BOX limited by the discontinuities.

Ex. 8.5.34: after shock formation



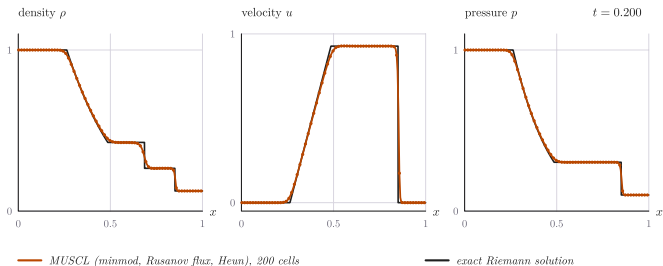
Observation: after shocks have formed, err_∞ does not converge and err_1 is $O(h)$ for both; MUSCL still gives a much smaller error constant (sharper shocks, better fans).

1. Argue why a linear monotonicity preserving piecewise linear reconstruction must impose vanishing slopes throughout.
2. Explain the meanings of “linear” in the phrase “non-linear linear reconstruction” (in the context of high-order finite volume methods for 1D conservation laws).
3. What will be the width of the stencil for the fully discrete evolution operator, when using two-point numerical fluxes, piecewise linear reconstruction based on minmod limiting, and an s -stage explicit RK-SSM for timestepping for a scalar conservation law in 1D?

8.6 *Outlook: systems of conservation laws*

8.6 Outlook: the Euler equations

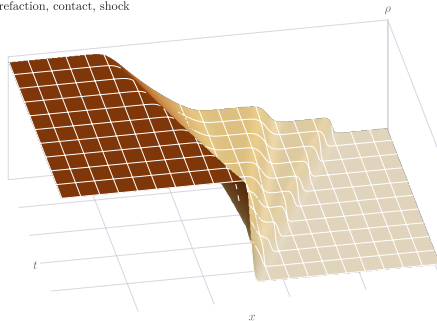
The Euler equations (8.1.48), a non-linear *system*: conservative finite volumes carry over componentwise ($\vec{\mu}_j \in \mathbb{R}^3$), with vector-valued numerical fluxes. The Riemann problem now has *three* waves.



Sod: $(\rho, u, p) = (1, 0, 1) / (0.125, 0, 0.1)$, $\gamma = 1.4$: rarefaction, contact discontinuity, shock. Minmod on primitive variables, Rusanov flux (cf. (8.3.36)), Heun.

The Sod problem over space-time

Euler, Sod: rarefaction, contact, shock



The MUSCL density $\rho(x, t)$ of the Sod shock tube as a surface: from left to right the rarefaction fan (a smooth ramp), the contact discontinuity and the shock, three creases fanning out from one point.

In this chapter about the numerical treatment of conservation laws you should have learned

- the general form of a scalar conservation law in one spatial dimension, and the balance law expressed by it;
- the notions of weak solutions, shock solutions, entropy conditions and entropy solutions;
- the general policy of constructing a conservative finite volume spatial semi-discretization;
- important consistent numerical flux functions, in particular the Godunov flux;
- the structure preservation inherent in conservative finite volume methods based on monotone numerical fluxes;
- the concept and significance of the CFL-condition for fully discrete conservative finite volume schemes;
- the construction of “high-order” spatial discretizations based on slope limited piecewise linear reconstruction.

9

Finite elements for the Stokes equations

9.1 Viscous fluid flow

Task: the velocity $\mathbf{v} = \mathbf{v}(x)$ of a fluid in a container $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, under an external force field $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$ (stationary flow). Incompressible fluids (Def. 7.1.7, Thm. 7.1.12):

$$\operatorname{div} \mathbf{v} = 0 \quad (9.1.1), \quad V := \{\mathbf{v} : \Omega \rightarrow \mathbb{R}^d \text{ continuous, } \operatorname{div} \mathbf{v} = 0\} \quad (9.1.2).$$

Reynolds number

$Re := \frac{\rho V L}{\mu}$ = inertia forces : viscous (friction) forces; ρ density, V mean velocity, L characteristic length, μ dynamic viscosity.

Re small for slow flows, tiny length scales (micro flows), or highly viscous fluids: neglect inertia, **creeping flow**.

Viscous fluids “stick to the walls”: **no-slip** boundary conditions $\mathbf{v} = 0$ on $\partial\Omega$ (9.1.3), and

$$V := \{\mathbf{v} : \Omega \rightarrow \mathbb{R}^d \text{ continuous, } \operatorname{div} \mathbf{v} = 0, \mathbf{v}|_{\partial\Omega} = 0\}. \quad (9.1.4)$$

An extremal principle: energy dissipation

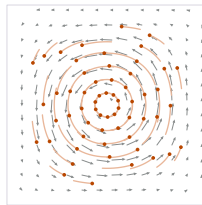
For stationary viscous flow the **energy dissipation** (kinetic energy $\rightarrow$ heat) is extremal.

Axiom: energy dissipation functional

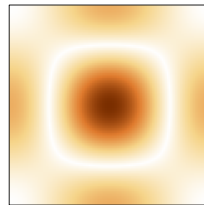
$$P_{\text{diss}}(\mathbf{v}) = \int_{\Omega} \mu \|\mathbf{curl} \mathbf{v}(x)\|^2 dx \quad [\text{W}], \quad (9.1.5)$$

(9.1.6): $\mathbf{curl} \mathbf{v} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$ ($d = 2$), the usual rotation for $d = 3$.

$\mathbf{curl} \mathbf{v}$ = density of eddies.



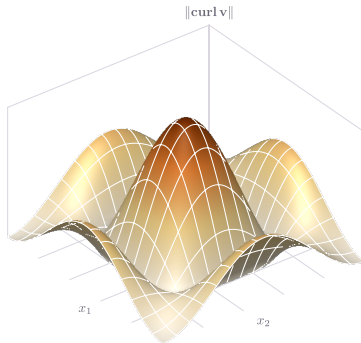
eddy field, $\text{div} \mathbf{v} = 0$



$\|\mathbf{curl} \mathbf{v}\|$: density of eddies

An eddy field (curl of $\sin^2(\pi x_1) \sin^2(\pi x_2)$), tracer particles; right: $\|\mathbf{curl} \mathbf{v}\|$.

Where the dissipation happens



$\|\text{curl } \mathbf{v}\|$ of the eddy field as a surface: the integrand of the dissipation functional (9.1.5) is largest in the vortex core and along the walls, where the no-slip condition (9.1.3) shears the flow.

Second law (creeping flow): maximization of energy dissipation (9.1.7)

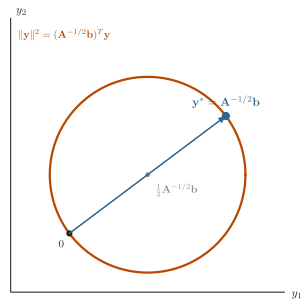
First law (power balance): $\int_{\Omega} \mu \|\mathbf{curl} \mathbf{v}\|^2 dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx$ (9.1.8)

$$\mathbf{v}^* = \operatorname{argmax} \left\{ \int_{\Omega} \mu \|\mathbf{curl} \mathbf{v}\|^2 : \mathbf{v} \in V, (9.1.8) \right\}, \quad (9.1.9)$$

a constrained optimization problem. **Finite-dimensional**

analogue: $\mathbf{x}^* = \operatorname{argmax}_{\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{b}^T \mathbf{x}} \mathbf{x}^T \mathbf{A} \mathbf{x}$ (9.1.10), $\mathbf{A}$ s.p.d.

With $\mathbf{y} = \mathbf{A}^{1/2} \mathbf{x}$: the farthest point from 0 on a sphere through 0, $\mathbf{y}^* = \mathbf{A}^{-1/2} \mathbf{b}$, so $\mathbf{x}^* = \mathbf{A}^{-1} \mathbf{b}$.



$\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{b}$ is also the solution of the quadratic minimization problem

$$\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{b}^T \mathbf{x} \quad (9.1.11)$$

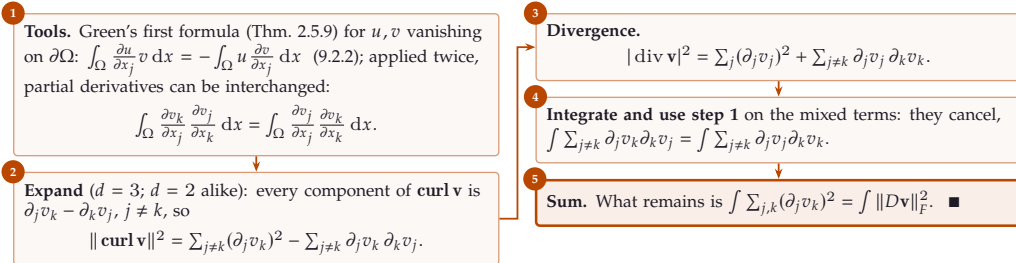
(cf. [8, Sect. 8.1.1], Sect. 2.2.3). For (9.1.9) this needs $(\mathbf{u}, \mathbf{v}) \mapsto \int \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v}$ to be positive definite (Lemma 9.2.1 below); the infinite-dimensional theory: variational calculus [11, Ch. 49].

Second equilibrium condition for viscous stationary flow, cf. (2.2.12), (2.2.24)

$$\mathbf{v}^* = \operatorname{argmin}_{\mathbf{v} \in V} \frac{1}{2} \int_{\Omega} \mu \|\mathbf{curl} \mathbf{v}\|^2 dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx. \quad (9.1.12)$$

9.2.1 Lemma 9.2.1: $-\Delta = \text{curl curl} - \text{grad div}$

Lemma 9.2.1. For $\mathbf{v} \in C^2(\Omega)$, $\mathbf{v}|_{\partial\Omega} = 0$: $\int_{\Omega} \|\text{curl } \mathbf{v}\|^2 dx + \int_{\Omega} |\text{div } \mathbf{v}|^2 dx = \int_{\Omega} \|D\mathbf{v}\|_F^2 dx$, $D\mathbf{v} := (\frac{\partial v_i}{\partial x_j})_{i,j}$, $\|\cdot\|_F$ the Frobenius norm.



A quadratic minimization problem on V

Since $\operatorname{div} \mathbf{v} = 0$, $\mathbf{v} = 0$ on $\partial\Omega$ for $\mathbf{v} \in V$ (9.1.4), Lemma 9.2.1 gives the equivalence

$$(9.1.12) \iff \mathbf{v}^* = \operatorname{argmin}_{\mathbf{v} \in V} \underbrace{\frac{1}{2} \int_{\Omega} \mu \|D\mathbf{v}\|_F^2 dx}_{=: \frac{1}{2} a(\mathbf{v}, \mathbf{v})} - \underbrace{\int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx}_{=: \ell(\mathbf{v})}, \quad (9.2.3)$$

a quadratic minimization problem (Def. 2.2.32). With μ const: $\int_{\Omega} \mu \|D\mathbf{v}\|_F^2 = \mu \sum_i \int_{\Omega} \|\mathbf{grad} v_i\|^2$. First Poincaré–Friedrichs inequality (Thm. 2.3.31): $\|\mathbf{v}\|_{L^2(\Omega)}^2 \leq \operatorname{diam}(\Omega)^2 \int_{\Omega} \|D\mathbf{v}\|_F^2 dx \Rightarrow a$ is positive definite (Def. 2.2.40).

Remark 9.2.4 (Decoupling of velocity components?) In components: $\operatorname{argmin}_{\mathbf{v} \in V} \sum_{i=1}^3 \left(\int_{\Omega} \frac{1}{2} \mu \|\mathbf{grad} v_i\|^2 dx - \int_{\Omega} f_i v_i dx \right)$ (9.2.5): three copies of (2.2.24)?! **No!** The constraint $\operatorname{div} \mathbf{v} = 0$ links the components: the crucial difference from scalar 2nd-order elliptic problems.

The largest spaces on which the forms are continuous (cf. (2.2.55), (3.2.4)):

$$\mathbf{H}_0^1(\operatorname{div} 0, \Omega) := \{\mathbf{v} \in (H_0^1(\Omega))^3 : \operatorname{div} \mathbf{v} = 0\}.$$

Linear variational problem (9.2.6)

Seek $\mathbf{v} \in \mathbf{H}_0^1(\operatorname{div} 0, \Omega)$ such that

$$\int_{\Omega} D\mathbf{v} : D\mathbf{w} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\operatorname{div} 0, \Omega), \quad A : B := \sum_{i,j} a_{ij} b_{ij}.$$

Assumption 5.1.2 (Poincaré–Friedrichs), 5.1.3 ($\mathbf{f} \in (L^2)^d$, Cauchy–Schwarz), 5.1.4 ($\mathbf{H}_0^1(\operatorname{div} 0, \Omega)$ closed in $\mathbf{H}^1$) $\Rightarrow$ Thm. 5.1.5: existence & uniqueness. **Remark 9.2.7.** Conforming divergence free FE bases exist only for degree ≥ 4 [9] $\Rightarrow$ move the constraint into the variational formulation.

9.2.2 Saddle point problem: Lagrange multipliers

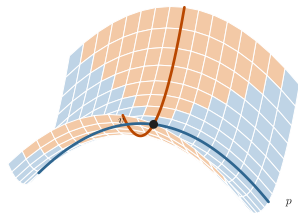
Rem. 9.2.8. U, Q Hilbert spaces, $J : U \rightarrow \mathbb{R}$ convex, differentiable, $B : U \rightarrow Q$ linear: $\mathbf{v}^* = \operatorname{argmin}_{B\mathbf{v}=0} J(\mathbf{v})$
(9.2.9). Lagrangian:

$$L(\mathbf{v}, p) := J(\mathbf{v}) + (p, B\mathbf{v})_Q, \quad \mathbf{v}^* = \operatorname{argmin}_{\mathbf{v} \in U} \sup_{p \in Q} L(\mathbf{v}, p), \quad (9.2.10)$$

($B\mathbf{v} \neq 0$ makes the sup $+\infty$); $p =$ **Lagrange multiplier**: a **saddle point problem**.

Lemma 9.2.11 (Necessary conditions) [11, Ch. 50]

$\mathbf{v}^*$ is the first component of a zero $(\mathbf{v}^*, p^*)$ of the derivative of L (9.2.12), since $L(\mathbf{v}^*, p) \leq L(\mathbf{v}^*, p^*) \leq L(\mathbf{v}, p^*)$ (9.2.13).



minimum along v

maximum along p

A saddle point: a minimum from the U -direction, a maximum from the Q -direction.

Computing the directional derivatives as in Sect. 1.3.1:

$$\langle DJ(\mathbf{v}^*), \mathbf{w} \rangle + (p^*, B\mathbf{w})_Q = 0 \quad \forall \mathbf{w} \in U, \quad (q, B\mathbf{v}^*)_Q = 0 \quad \forall q \in Q, \quad (9.2.14)$$

a **variational saddle point problem**. Special case: quadratic $J(\mathbf{v}) := \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - \ell(\mathbf{v})$, a s.p.d.,
 $\langle DJ(\mathbf{v}^*), \mathbf{w} \rangle = a(\mathbf{v}^*, \mathbf{w}) - \ell(\mathbf{w})$ (2.4.9):

Linear variational saddle point problem

Seek $\mathbf{v}^* \in U, p^* \in Q$:

$$a(\mathbf{v}^*, \mathbf{w}) + (p^*, B\mathbf{w})_Q = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in U, \quad (q, B\mathbf{v}^*)_Q = 0 \quad \forall q \in Q. \quad (9.2.15)$$

Rigorous treatment in Banach spaces: [11, Ch. 49 & 50]; finite-dimensional setting [8, Sect. 6.4.1].

Adapt to (9.2.6): $U = \mathbf{H}_0^1(\Omega)$, $Q = L^2(\Omega)$, $B = \operatorname{div} : U \rightarrow Q$, $J(\mathbf{v}) = \frac{1}{2} \int \mu \|D\mathbf{v}\|^2 - \int \mathbf{f} \cdot \mathbf{v}$:

$$L(\mathbf{v}, p) = \frac{1}{2} \int_{\Omega} \mu \|D\mathbf{v}\|_F^2 dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx + \int_{\Omega} \operatorname{div} \mathbf{v} p dx. \quad (9.2.16)$$

Remark 9.2.17. $\int_{\Omega} \operatorname{div} \mathbf{v} dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} dS = 0$: p is unique only up to a constant (cf. the Neumann problem, Rem. 2.9.14) $\Rightarrow p \in L_*^2(\Omega) := \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}$ (9.2.18).

Stokes problem: variational saddle point problem for viscous flow

Seek velocity $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and Lagrange multiplier $p \in L_*^2(\Omega)$ (= **pressure**, $[p] = \text{N m}^{-2}$):

$$\begin{aligned} \int_{\Omega} \mu D\mathbf{v} : D\mathbf{w} dx + \int_{\Omega} \operatorname{div} \mathbf{w} p dx &= \int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx & \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega), \\ \int_{\Omega} \operatorname{div} \mathbf{v} q dx &= 0 & \forall q \in L_*^2(\Omega). \end{aligned} \quad (9.2.19)$$

No differential constraints in the trial/test spaces!

Theorem 9.2.20: existence and uniqueness

Theorem 9.2.20 (Existence and uniqueness of weak solutions of Stokes problem)

The linear variational saddle point problem (9.2.19) has a unique solution.

- 1 **Velocity.** $a(\mathbf{v}, \mathbf{w}) = \int \mu D\mathbf{v} : D\mathbf{w}$ is an inner product on $\mathbf{H}_0^1(\Omega)$; a -orthogonal decomposition $\mathbf{H}_0^1(\Omega) = \mathbf{H}_0^1(\operatorname{div} 0, \Omega) \oplus V^\perp$. Test with $\mathbf{w} \in \mathbf{H}_0^1(\operatorname{div} 0, \Omega)$: (9.2.6) has a unique solution $\Rightarrow$ unique $\mathbf{v}$ (then test with $\mathbf{w} \in V^\perp$).
- 2 **Uniqueness of p .** For $\mathbf{f} = 0$ test the first equation with $\mathbf{w} \in V^\perp$, $\operatorname{div} \mathbf{w} = p$ (Thm. 9.2.21): $\int p^2 = 0$, so $\|p\|_{L^2} = 0$, $p = 0$.
- 3 **Existence of p** from the Riesz representation theorem and Thm. 9.2.21 (functional analysis, not elaborated). The ideas carry over to the discrete setting. ■

Theorem 9.2.21 [2, Thm. 5.3]: $\exists C(\Omega): \forall q \in L_*^2(\Omega) \exists \mathbf{v} \in \mathbf{H}_0^1(\Omega), q = \operatorname{div} \mathbf{v}, \|\mathbf{v}\|_{H^1} \leq C\|q\|_{L^2}$.

Remark 9.2.22: enforcing zero mean

(9.2.18) is another constraint: Lagrange multipliers deal with it, too. Neumann problem (2.9.16) on $H_*^1(\Omega)$ (2.9.15); enforce $\int_{\Omega} v \, dx = 0$ by a scalar multiplier $p \in \mathbb{R}$: $L(v, p) = J(v) + p \int_{\Omega} v \, dx$. Seek $u \in H^1(\Omega)$, $p \in \mathbb{R}$:

$$\begin{aligned} \int_{\Omega} \kappa \mathbf{grad} u \cdot \mathbf{grad} v \, dx + p \int_{\Omega} v \, dx &= \int_{\Omega} f v \, dx \quad \forall v, \\ \int_{\Omega} u \, dx &= 0. \end{aligned} \quad (9.2.23)$$

Stokes problem with pressure normalization (9.2.24)

Seek $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $p \in L^2(\Omega)$, $\lambda \in \mathbb{R}$:

$$\begin{aligned} \int \mu \nabla \mathbf{v} : \nabla \mathbf{w} + \int \operatorname{div} \mathbf{w} p &= \int \mathbf{f} \cdot \mathbf{w} \quad \forall \mathbf{w}, \\ \int \operatorname{div} \mathbf{v} q + \lambda \int q &= 0 \quad \forall q \in L^2(\Omega), \\ \int p &= 0. \end{aligned}$$

9.2.3 The Stokes system

1

Integrate by parts (Green, Thm. 2.5.9), smooth $(\mathbf{v}, p)$, μ const, $\mathbf{w} = 0$ on $\partial\Omega$:

$$\int \mu \nabla \mathbf{v} : \nabla \mathbf{w} = -\mu \sum_i \int \Delta v_i w_i,$$

$$\int \operatorname{div} \mathbf{w} p = -\int \operatorname{grad} p \cdot \mathbf{w}.$$



2

Stokes system (9.2.25), $\Delta =$ vector Laplacian:

$$-\mu \Delta \mathbf{v} - \operatorname{grad} p = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,$$

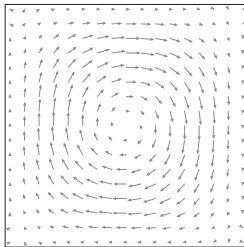
$$\int_{\Omega} p = 0, \quad \mathbf{v} = 0 \text{ on } \partial\Omega.$$

Remark 9.2.26 (Pressure Poisson equation).

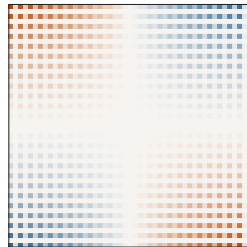
Apply div to (9.2.25): $-\mu \Delta(\operatorname{div} \mathbf{v}) + \Delta p = \operatorname{div} \mathbf{f}$, so $\Delta p = \operatorname{div} \mathbf{f}$. Apparently $d+1$ Poisson equations: first p , then $-\Delta v_i = f_i + \frac{\partial p}{\partial x_i}$, $v_i = 0$ on $\partial\Omega$. **But** the manipulations need smooth solutions (not guaranteed), and we cannot solve a “Poisson equation” for p without boundary conditions, which are not available!

9.3 Ex. 9.3.1: naive finite differences

(9.2.25) on $]0, 1[^2$, $\mu = 1$, stirring force $\mathbf{f} = \cos(\pi x_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; tensor product grid, unknowns $v_{1,ij}, v_{2,ij}, p_{ij}$ at (ih, jh) ; zero boundary values for v_1, v_2, p ; 5-point stencil for $-\Delta$ (4.1.3); central differences for $\text{grad } p$, $\frac{\partial p}{\partial x_1} \approx \frac{1}{2h}(p_{i+1,j} - p_{i-1,j})$.



velocity (v_1, v_2) , $N = 50$



pressure p : odd-even oscillations

Physically meaningless solution, marred by massive spurious oscillations of the pressure. Lesson: discretizing saddle point problems can be tricky!

9.3.1 Galerkin discretization

Shorthand for (9.2.19), cf. (9.2.15), with $a(\mathbf{v}, \mathbf{w}) := \mu \int_{\Omega} D\mathbf{v} : D\mathbf{w}$, $b(\mathbf{v}, q) := \int_{\Omega} \operatorname{div} \mathbf{v} q$ (9.3.4):

$$\mathbf{v} \in U := \mathbf{H}_0^1(\Omega), p \in Q := L_*^2(\Omega) : \quad a(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = \ell(\mathbf{w}) \quad \forall \mathbf{w} \in U, \quad b(\mathbf{v}, q) = 0 \quad \forall q \in Q. \quad (9.3.3)$$

Replace U, Q by finite dimensional $U_N \subset U, Q_N \subset Q$ (9.3.5); bases $\{\mathbf{b}_N^j\}_{j=1}^N, \{\beta_N^i\}_{i=1}^M, \mathbf{v}_N = \sum_j v_j \mathbf{b}_N^j, p_N = \sum_j \pi_j \beta_N^j$ (9.3.11):

$(N+M) \times (N+M)$ symmetric indefinite system

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{\pi} \end{pmatrix} = \begin{pmatrix} \vec{\varphi} \\ \mathbf{0} \end{pmatrix} \quad (9.3.6)$$

$$\mathbf{A} = (a(\mathbf{b}_N^j, \mathbf{b}_N^i))_{i,j} \in \mathbb{R}^{N,N} \quad (9.3.8),$$

$$\mathbf{B} = (b(\mathbf{b}_N^j, \beta_N^i))_{i,j} \in \mathbb{R}^{M,N} \quad (9.3.9),$$

$$\vec{\varphi} = (\ell(\mathbf{b}_N^j))_j \in \mathbb{R}^N \quad (9.3.10).$$

Issue. Existence, uniqueness and stability of solutions of discrete variational saddle point problems *cannot* be inferred from these properties for the continuous saddle point problem (Thm. 9.2.20), unlike for linear variational problems with s.p.d. bilinear forms (Thm. 3.2.9)!

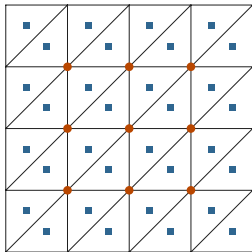
A simple consideration:

$$M > N \Rightarrow \text{Kern } \mathbf{B}^T \neq \{0\} \Rightarrow \text{non-uniqueness of } p_N,$$

so $\dim U_N \geq \dim Q_N$ is a **necessary condition** for a unique pressure solution of (9.3.5). Some “natural” finite element Galerkin schemes for (9.2.19) fail to meet this condition.

Ex. 9.3.12: the unstable P1-P0 pair

$S_p^{-1}(\mathcal{M})$: discontinuous piecewise polynomials of degree p , the natural FE spaces in $L^2(\Omega)$.
 $U_N := (S_{1,0}^0(\mathcal{M}))^2$, $Q_N := S_0^{-1}(\mathcal{M}) \cap L_*^2(\Omega)$ (piecewise constants) on a regular triangular mesh:



$K = 4$ cells per direction

$$\dim U_N = 2(K-1)^2 = 18$$

(two velocity components per interior vertex)

$$\dim Q_N = 2K^2 - 1 = 31$$

(one constant per triangle, zero mean)

$$\dim Q_N > \dim U_N: \text{Kern } \mathbf{B} \neq \{0\}$$

A singular system (9.3.6): the pressure is polluted by “noise” from Kern $\mathbf{B}^T$.

Ex. 9.3.13: checkerboard instability

Uniform tensor product mesh, bilinear U_N ,
piecewise constant Q_N :
 $\dim U_N = 2(K-1)^2 > \dim Q_N = K^2 - 1$ for
 $K \geq 4$.

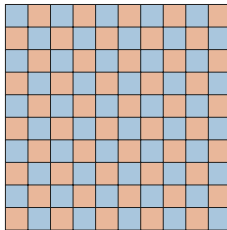
1

Gauss on each C_i for $\mathbf{b}_{N,1}^p = b_N^p(\frac{1}{0})$ (b_N^p the tent function at $\mathbf{p}$): $\mathbf{b}_N^p \perp$ normals on e_2, e_4 :

$$\int_{\Omega} \operatorname{div} \mathbf{b}_{N,1}^p p_N = (p_1 - p_2) \int_{e_1} b_N^p + (p_3 - p_4) \int_{e_3} b_N^p$$

2

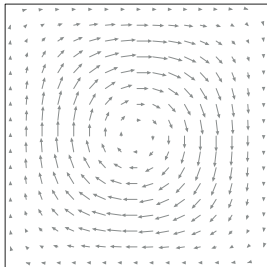
Edge integrals = $\frac{1}{2} = \frac{1}{2}(p_1 - p_2 + p_3 - p_4)$;
likewise $\frac{1}{2}(p_1 + p_2 - p_3 - p_4)$ for $\mathbf{b}_{N,2}^p$. Both vanish
for $p_1 = p_3 = 1, p_2 = p_4 = -1$ (9.3.14).



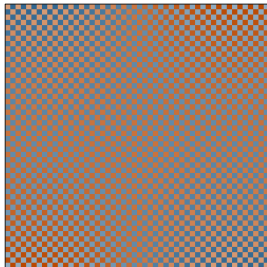
translation invariance: the checkerboard mode lies in $\operatorname{Kern} \mathbf{B}^T$: invisible to every discrete divergence

Translation invariance: the ± 1 checkerboard pressure lies in $\operatorname{Kern} \mathbf{B}^T$, a one-dimensional space of spurious modes.

Ex. 9.3.15: quadrilateral P1-P0



velocity: fine



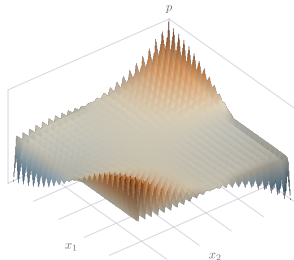
pressure: marred by the checkerboard mode

BVP (9.2.25) of Ex. 9.3.1 on $]0, 1[^2$,
augmented system (9.2.24),
 $N = 50$, direct sparse solver.

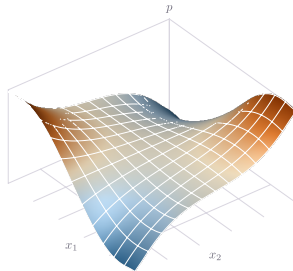
Observation:

- the pressure solution is marred by the checkerboard mode;
- the computed velocity field is fine!

Pressure instability in 3D



naive FD (Ex. 9.3.1)

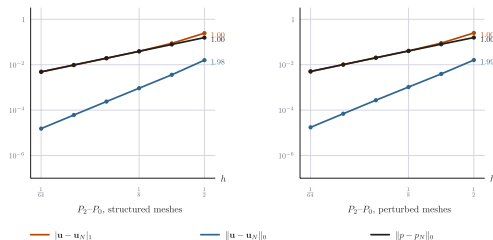


Taylor-Hood (Sect. 9.4)

Left: the pressure of the naive finite difference scheme (Ex. 9.3.1) as a surface: odd-even spikes, physically meaningless. Right: the Taylor-Hood pressure for the same stirring force (Sect. 9.4): smooth, the price of stability is paid in the velocity space.

9.3.2 Stable discretization: Ex. 9.3.17

Heuristic: $\text{div } \mathbf{v}_N$ must be “large enough to fix the pressure” $\Rightarrow$ larger velocity spaces: raise the polynomial degree. **P2-P0**: $U_N = (S_{2,0}^0(\mathcal{M}))^2$, $Q_N = S_0^{-1}(\mathcal{M}) \cap L_*^2(\Omega)$; $\mathbf{v} = \begin{pmatrix} \cos(\frac{\pi}{2}(x_1+x_2)) \\ -\cos(\frac{\pi}{2}(x_1+x_2)) \end{pmatrix}$, $p = \sin(\frac{\pi}{2}(x_1 - x_2))$, regular refinement and randomly perturbed meshes:



Raising the polynomial degree has cured the instability! $\|\mathbf{v} - \mathbf{v}_N\|_1, \|p - p_N\|_0 = O(h)$, $\|\mathbf{v} - \mathbf{v}_N\|_0 = O(h^2)$.

Stability: Definition 9.3.21 and Remark 9.3.23

Stability for linear problems: solution $\leq C \cdot$ data, in the norms of (9.2.19): $\|\cdot\|_a := a(\cdot, \cdot)^{1/2} \sim \|\cdot\|_{H^1}$ for $\mathbf{v}$, $\|\cdot\|_{L^2}$ for p .

Definition 9.3.21 (Stable finite element pair)

$U_N \subset \mathbf{H}_0^1(\Omega)$, $Q_N \subset L_*^2(\Omega)$ is a stable pair, if the solution of (9.3.5) satisfies

$$|\ell(\mathbf{w}_N)| \leq C_\ell \|\mathbf{w}_N\|_a \quad \forall \mathbf{w}_N \implies \exists C > 0 : \|\mathbf{v}_N\|_a + \|p_N\|_{L^2} \leq CC_\ell,$$

C depending only on Ω , μ , and the shape regularity of $\mathcal{M}$. ((9.3.22): the bound on ℓ)

1

Discrete kernel. $N(b_N) := \{\mathbf{w}_N \in U_N : b(\mathbf{w}_N, q_N) = 0 \quad \forall q_N\}$ (9.3.24); by the 2nd equation $\mathbf{v}_N \in N(b_N)$.

2

Remark 9.3.23. Test the 1st equation with $\mathbf{w}_N = \mathbf{v}_N$: $\|\mathbf{v}_N\|_a^2 = \ell(\mathbf{v}_N) \leq C_\ell \|\mathbf{v}_N\|_a$: **perfect stability** of any velocity Galerkin solution (explains Ex. 9.3.15).

Thm. 9.3.28: the inf-sup condition

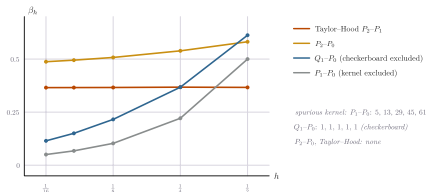
Rem. 9.3.25. Goal $\|p_N\|_{L^2} \leq C \sup_{\mathbf{w}_N} \frac{\ell(\mathbf{w}_N)}{\|\mathbf{w}_N\|_a}$
(9.3.26) follows once $b(\mathbf{w}_N, p_N) = g(\mathbf{w}_N) \forall \mathbf{w}_N$
implies $\|p_N\| \leq C \sup \frac{|g(\mathbf{w}_N)|}{\|\mathbf{w}_N\|_a}$ (9.3.27).

Theorem 9.3.28 (inf-sup condition)

U_N, Q_N is a stable pair, if $\exists \beta > 0$ (depending only on Ω and shape regularity):

$$\sup_{\mathbf{w}_N \in U_N} \frac{|b(\mathbf{w}_N, q_N)|}{\|\mathbf{w}_N\|_a} \geq \beta \|q_N\|_{L^2} \quad \forall q_N.$$

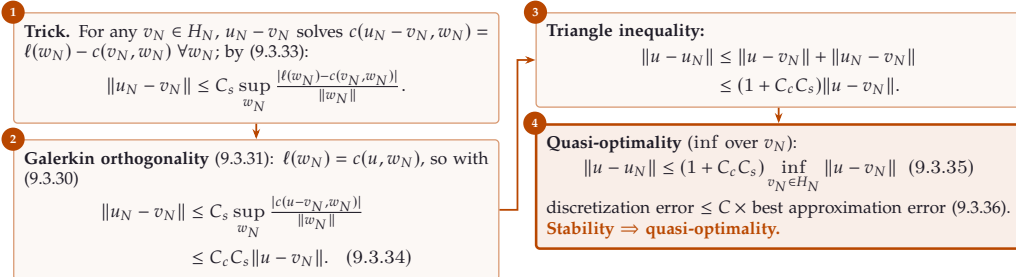
LBB condition: the linchpin of the analysis [6].



Computed β_h (from $\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$ and the pressure mass matrix): bounded below for P2-P0 and Taylor-Hood, $O(h)$ for Q_1-P_0 .

9.3.3 Convergence: quasi-optimality

H normed space, $c : H \times H \rightarrow \mathbb{R}$ bilinear, continuous: $|c(u, v)| \leq C_c \|u\| \|v\|$ (9.3.30); $u \in H$:
 $c(u, v) = \ell(v) \forall v$ (9.3.31); Galerkin $u_N \in H_N$ (9.3.32), stable: $\|u_N\| \leq C_s \sup_{w_N} \frac{|\ell(w_N)|}{\|w_N\|}$ (9.3.33).



Theorem 9.3.38 and Example 9.3.39

Apply to (9.2.19): $H := \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$, $c\left(\begin{pmatrix} \mathbf{v} \\ p \end{pmatrix}, \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix}\right) := a(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) + b(\mathbf{v}, q)$ (9.3.37),
 $H_N := U_N \times Q_N$:

Theorem 9.3.38 (Convergence of stable FE for Stokes problem)

If U_N, Q_N is a stable pair (Def. 9.3.21), the Galerkin solution satisfies

$$\|\mathbf{v} - \mathbf{v}_N\|_{H^1(\Omega)} + \|p - p_N\|_{L^2(\Omega)} \leq C \left(\inf_{\mathbf{w}_N \in U_N} \|\mathbf{v} - \mathbf{w}_N\|_{H^1(\Omega)} + \inf_{q_N \in Q_N} \|p - q_N\|_{L^2(\Omega)} \right),$$

C depending only on Ω , μ and the shape regularity of the mesh.

Example 9.3.39. For P2-P0 and smooth $\mathbf{v}, p$: $\inf_{\mathbf{w}_N \in S_{2,0}^0} \|\mathbf{v} - \mathbf{w}_N\|_{H^1} \leq Ch^2 \|\mathbf{v}\|_{H^3}$ (Thm. 5.3.56), but $\inf_{q_N \in S_0^{-1}} \|p - q_N\|_{L^2} \leq Ch \|p\|_{H^1}$: the **larger best approximation error of the pressure space dominates** $\Rightarrow$ the observed $O(h)$.

9.4 The Taylor–Hood element

A: the ultimate cure: a velocity space large enough, very large, to play safe.

B: a large space means a large linear system, high cost.

A: never mind, it buys good accuracy!

Remark 9.4.1. By Thm. 9.3.38 the error is the *sum* of the best approximation errors of $\mathbf{v}$ in U_N and p in Q_N : excellent approximation of one alone does not help (cf. the method of lines, Thms. 6.1.96, 6.2.57).

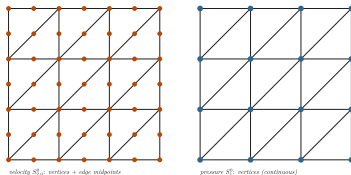
Guideline for the Galerkin spaces for Stokes

- ❶ (U_N, Q_N) must be stable (Def. 9.3.21);
- ❷ U_N should give the same algebraic rate of the H^1 -best approximation error as Q_N in L^2 ;
- ❸ with as few degrees of freedom as possible.

$(S_{2,0}^0, S_0^{-1})$ fails ❷: the velocity space converges faster than the pressure space (Ex. 9.3.39).

A stable, perfectly balanced pair

Taylor–Hood: $\mathcal{M}$ triangular/tetrahedral or rectangular/hexahedral (may be hybrid, Sect. 3.4.1); velocity $U_N := S_{2,0}^0(\mathcal{M}) \subset \mathbf{H}_0^1(\Omega)$, pressure $Q_N := S_1^0(\mathcal{M})$ (continuous):



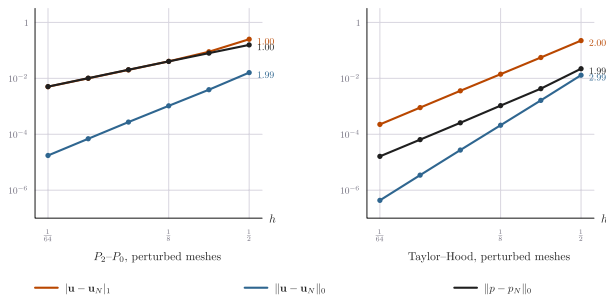
Balanced: both best approximation errors are $O(h^2)$ (Thms. 5.3.56, 5.3.38).

Theorem 9.4.2 (Stability and convergence of Taylor–Hood) [10]

Stable pair, and $\|\mathbf{v} - \mathbf{v}_N\|_{H^1} + \|p - p_N\|_{L^2} \leq Ch^2 (\|\mathbf{v}\|_{H^3} + \|p\|_{H^2})$.

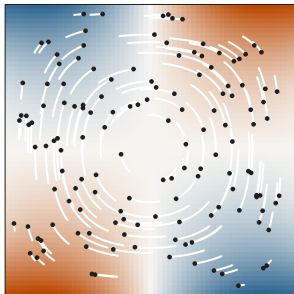
Example 9.4.3: convergence of Taylor–Hood

Stokes problem of Ex. 9.3.17, perturbed triangular meshes; P2-P0 (left) against Taylor–Hood (right):

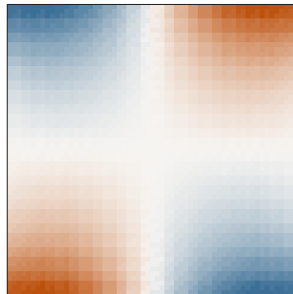


Observation: $\|\mathbf{v} - \mathbf{v}_N\|_{H^1} = O(h^2)$, $\|\mathbf{v} - \mathbf{v}_N\|_{L^2} = O(h^3)$, $\|p - p_N\|_{L^2} = O(h^2)$, as predicted by Thm. 9.4.2 (the script reports slopes 2.15 and 3.11).

The stirring problem, solved stably



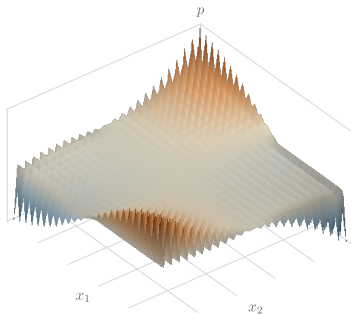
Taylor-Hood: pressure and particle paths



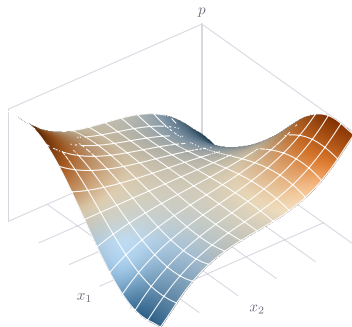
P_2-P_0 : piecewise constant pressure

The force $\mathbf{f} = \cos(\pi x_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of Ex. 9.3.1 with Taylor-Hood ($K = 24$): a smooth pressure and the eddy it drives (particle paths). Right: P_2-P_0 , stable but only piecewise constant.

- **Lecture I:** conservation laws, characteristics, weak and entropy solutions.
- **Lecture II:** conservative finite volumes, numerical fluxes, monotone schemes, CFL, stability.
- **Lecture III:** slope limiting and MUSCL; systems.
- Stokes: a constrained minimization $\rightarrow$ a **saddle point problem**, pressure = Lagrange multiplier.
- Discrete saddle points need the **inf-sup condition**; P1-P0 fails (checkerboard), P2-P0 and **Taylor–Hood** are stable, Taylor–Hood is balanced.
- Stability $\Rightarrow$ quasi-optimality.



naive FD (Ex. 9.3.1)



Taylor-Hood (Sect. 9.4)

Thank you.
Johannes Herter