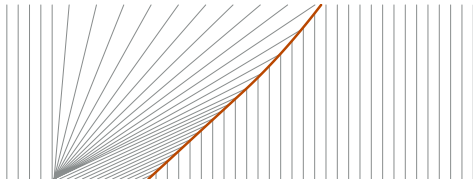


Conservation Laws I

Models, characteristics and entropy solutions (Sections 8.1–8.2)

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Three lectures on Chapters 8 and 9

Lecture I (today)

8.1 Conservation laws: examples

8.2 Scalar conservation laws in 1D: characteristics, weak solutions, jumps, Riemann problem, entropy solutions

Lecture II

8.3 Conservative finite volumes: conservation form, numerical fluxes, monotone schemes

8.4 Timestepping: CFL, stability, convergence

Lecture III

8.5 Higher-order conservative schemes: reconstruction, slope limiting, MUSCL

8.6 Outlook: systems
9 Finite elements for the Stokes equations

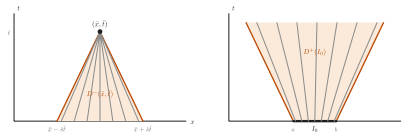
Thread. **Conservation** of physical quantities (mass, momentum, energy) and their **transport**; new compared with Chapter 7: the transport velocity depends on the conserved quantities themselves $\Rightarrow$ non-linear models.

Setting

Spatial domain $\Omega = \mathbb{R}^d$ (unbounded!)
 $\Rightarrow$ Cauchy problems are pure *initial value problems* (no boundary values).

Why do we restrict ourselves to Cauchy problems?

- 1 **Finite speed of propagation** (Thm. 8.2.43): for compactly supported u_0 boundaries do not affect the solution for some time, cf. the wave equation (6.2.21).
- 2 **No spatial boundary**: no boundary conditions (very intricate for conservation laws, Rem. 8.2.6).



Domain of dependence and domain of influence:
information travels at most at speed $\dot{s}$.

8.1 *Conservation laws: examples*

8.1.1 Linear advection

(8.1.1) Heat transport in a moving fluid. Cauchy problem for the linear transport (advection) equation, cf. Sect. 7.1.4, (7.1.16):

$$\frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\mathbf{v}(x, t)(\rho u)) = f(x, t) \quad \text{in } \tilde{\Omega} := \mathbb{R}^d \times]0, T[, \quad (8.1.2)$$

$$u(x, 0) = u_0(x) \quad \text{for all } x \in \mathbb{R}^d \quad (\text{initial conditions}). \quad (8.1.3)$$

$u = u(x, t) \hat{=}$ temperature, $\rho > 0 \hat{=}$ heat capacity, $\mathbf{v} = \mathbf{v}(x, t) \hat{=}$ prescribed locally Lipschitz-continuous velocity field, $\mathbf{v} : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^d$.

(8.1.2) = linear scalar conservation law

Conserved quantity: thermal energy (density) ρu .

Convention. div acts on the spatial variable only: $(\operatorname{div} \mathbf{f})(x, t) := \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_d}{\partial x_d}$.

Assume constant heat capacity $\rho \equiv 1$, no sources $f \equiv 0$, a stationary velocity field $\mathbf{v} = \mathbf{v}(x)$: rescaled initial value problem written in conserved variables

$$\frac{\partial u}{\partial t} + \operatorname{div}(\mathbf{v}(x)u) = 0 \quad \text{in } \tilde{\Omega} := \mathbb{R}^d \times]0, T[, \quad u(x, 0) = u_0(x) \quad \text{for all } x \in \mathbb{R}^d. \quad (8.1.4)$$

A general solution formula exists, based on the **flow map** $\Phi : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ induced by $\mathbf{v}$ (cf. (7.1.3)):

$$\frac{\partial \Phi}{\partial t}(x, t) = \mathbf{v}(\Phi(x, t)) \quad \text{in } \mathbb{R}^d \times \mathbb{R}, \quad \Phi(x, 0) = x \quad \forall x \in \mathbb{R}^d. \quad (8.1.5)$$

By existence and uniqueness for ODE initial value problems [33, Thm. 11.1.32] Φ is well defined for (locally) Lipschitz continuous $\mathbf{v}$, and it satisfies

$$\Phi(\Phi(x, t), -t) = x \quad \text{for all } x \in \mathbb{R}^d. \quad (8.1.6)$$

The solution formula for linear advection

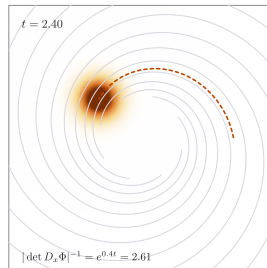
Theorem 8.1.7 (Solution of linear advection problem)

The solution of (8.1.4) is given by

$$u(x, t) = |\det(D_x \Phi)(x, t)|^{-1} u_0(\Phi(x, -t)), \quad (8.1.8)$$

$(x, t) \in \mathbb{R}^d \times \mathbb{R}$, where $D_x \Phi$ is the Jacobian of the flow map.

Reading. $\Phi(x, -t)$ traces x back to its origin; $|\det D_x \Phi|^{-1}$ accounts for compression of volume along the flow ($= 1$ if $\operatorname{div} \mathbf{v} = 0$).



$$\mathbf{v}(x) = \begin{pmatrix} -0.2 & -1 \\ 1 & -0.2 \end{pmatrix} x, \operatorname{div} \mathbf{v} = -0.4: \\ |\det D_x \Phi|^{-1} = e^{0.4t}.$$

Example 8.1.9: constant advection in 1D

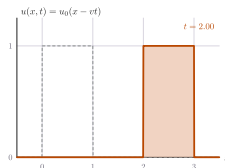
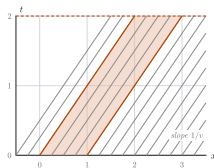
$d = 1$, $\Omega = \mathbb{R}$, constant velocity v :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(vu) = 0 \text{ in } \tilde{\Omega}, \quad u(x, 0) = u_0(x). \quad (8.1.10)$$

1D transport equation (7.3.7); by (7.3.11), a special case of Thm. 8.1.7:

$$u(x, t) = u_0(x - vt), \quad x \in \mathbb{R}, \quad 0 \leq t < T, \quad (8.1.11)$$

the initial data “travelling” with velocity v .



Remark 8.1.12 (Discontinuous solutions of advection equations). (8.1.11) needs $u_0 \in C^1$ classically, yet Thm. 8.1.7 makes sense for *discontinuous* u_0 : a “weaker” concept of solution is required (Sect. 8.2.3), cf. variational solutions (Rem. 1.3.47), d’Alembert (6.2.22).

Rem. 8.1.13: boundary conditions

Scalar linear advection IBVP on a bounded domain
 $\Omega \subset \mathbb{R}^d$, cf. Sects. 7.2.1, 7.3.2:

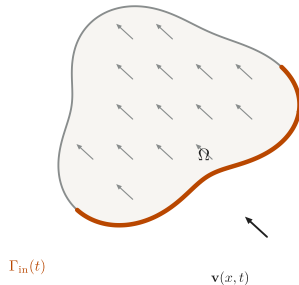
$$\frac{\partial u}{\partial t} + \operatorname{div}(\mathbf{v}(x, t)u) = f(x, t) \text{ in } \Omega \times]0, T[, \quad (8.1.14)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (8.1.15)$$

Boundary conditions $u(x, t) = g(x, t)$ on $\Gamma_{\text{in}}(t) \times]0, T[$
can be imposed on the **inflow boundary**

$$\Gamma_{\text{in}}(t) := \{x \in \partial\Omega : \mathbf{v}(x, t) \cdot \mathbf{n}(x) < 0\}. \quad (8.1.16)$$

Γ_{in} can change with time! **Bottom line:** meaningful
boundary conditions need the local and current
direction of transport.



8.1.2 Traffic modeling: the particle model



Non-stationary traffic flow on a single long highway lane (e.g. bypasses of long construction sites) [7].
Simplifying modeling assumptions (not quite matching reality):

- identical cars and behavior of drivers (8.1.17)

- uniformity of road conditions (8.1.18)

- speed of a car determined only by (its distance from) the car in front (8.1.19)

8.1.2.1 Particle model (agent based model): track a finite number of individual cars over $[0, T]$; semi-discrete, still continuous in time. Key state parameter of a car: its position,

$$x_i(t) \hat{=}\text{ position of the } i\text{-th car at time } t, \quad i = 1, \dots, N, \quad x_i(t) < x_{i+1}(t);$$

configuration space $\mathbb{R}^N$. The curve $t \mapsto x_i(t)$ in the x - t plane is the **trajectory** of the i -th car.

(8.1.20) Velocity model

Optimal velocity model

$$\dot{x}_i(t) = v_{\text{opt}}(\Delta x_i), \quad (8.1.21)$$

$$v_{\text{opt}}(\Delta x) = v_{\text{max}} \left(1 - \frac{\Delta_0}{\Delta x} \right), \quad (8.1.22)$$

$\Delta x_i(t) = x_{i+1}(t) - x_i(t) > 0$, $i = 1, \dots, N - 1$; $\Delta_0 \hat{=}$ length of a car = distance of cars in a bumper-to-bumper traffic jam.

By (8.1.19): each car drives as fast as possible under safety constraints.

(8.1.21) + (8.1.22): an ODE on $\mathbb{R}^N$, made well-posed by initial conditions

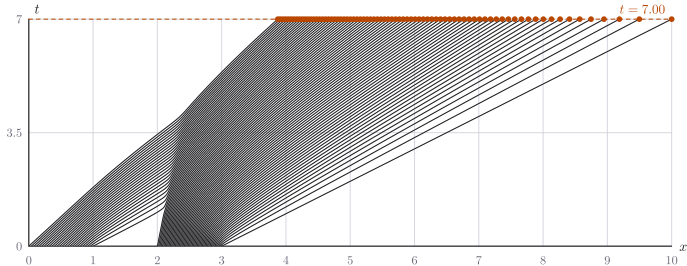
$$x_i(0) = x_{i,0} \in \mathbb{R}, \quad x_{i,0} \leq x_{i+1,0} - \Delta_0. \quad (8.1.23)$$

Obviously (why?) $x_i(t) \leq x_{i+1}(t) - \Delta_0$.

Remark 8.1.24 (Acceleration based traffic modeling). The speed of a car results from accelerating and braking: $\ddot{x}_i(t) = F(\Delta x_i(t), \Delta v_i(t))$, $\Delta v_i = \dot{x}_{i+1} - \dot{x}_i$ (8.1.25). Models of this type are popular in practice.

Exp. 8.1.26: particle simulation of traffic

$v_{\max} = 1$ after rescaling; two clusters of evenly spaced cars, $x_0 = [0 : \frac{2}{k} : 1, 2 : \frac{1}{k} : 3]$, $k = 50$ (77 cars), $\Delta_0 = 0.9/k = 0.018$.



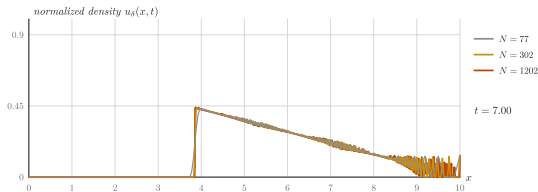
Trajectories $t \mapsto x_i(t)$. The clusters merge and dissolve as cars “escape” to the right; fan-shaped patterns emerge.

(8.1.29) Extraction of macroscopic quantities

Continuum model: **macroscopic quantities**, detached from individual cars, by averaging. The (normalized) density:

$$u_\delta(x, t) := \frac{\Delta_0}{2\delta} \# \{i \in \{1, \dots, N\} : x - \delta \leq x_i(t) < x + \delta\}, \quad (8.1.30)$$

$\delta > 0$ the averaging length; $0 \leq u_\delta \leq 1$. **Exp. 8.1.31:** $k = 50, 200, 800$, $\Delta_0 = 0.9/k$, $\delta = 3.33/k$.



Striking observation: for $N \rightarrow \infty$, $\Delta_0 \sim N^{-1}$, $\delta \sim N^{-1}$ the u_δ approach a *limit density*. (Triangular window of the same width.)

8.1.2.2 Continuum traffic model

Remark 8.1.32 (Suitability of macroscopic models for traffic flow). The limit $N \rightarrow \infty$ is commonly denounced as dubious for traffic [7, Sect. 2.3]. Nevertheless it yields a qualitatively correct picture and an important model problem for scalar non-linear conservation laws.

Ingredients: $\Omega = \mathbb{R}$ (Cauchy problem), normalized density $u : \Omega \times [0, T] \rightarrow [0, 1]$ as in (8.1.30), optimal velocity model (8.1.22).

(8.1.33) Neither Δx_i nor $\dot{x}_i(t)$ is a macroscopic quantity. Macroscopic velocity by spatial averaging of the velocities of cars:

$$v_\delta(x, t) = \frac{\sum_{i \in U_\delta(x)} \dot{x}_i(t)}{\#U_\delta(x)}, \quad U_\delta(x) := \{i \in \{1, \dots, N\} : x - \delta \leq x_i(t) < x + \delta\}, \quad (8.1.34)$$

and the (normalized) **flux of cars**

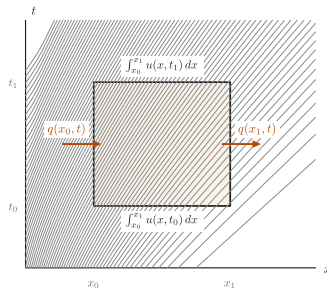
$$q_\delta(x, t) = u_\delta(x, t) v_\delta(x, t) \approx \text{no. of cars passing } x \text{ per unit time around } t. \quad (8.1.35)$$

Approximate balance law

“Conservation of cars” in a space-time box
 $[x_0, x_1] \times [t_0, t_1]$:

$$\underbrace{\int_{x_0}^{x_1} u_\delta(x, t_1) dx - \int_{x_0}^{x_1} u_\delta(x, t_0) dx}_{\text{change of no. of cars on } [x_0, x_1] \text{ in } [t_0, t_1]} \\ \approx \underbrace{\int_{t_0}^{t_1} q_\delta(x_0, t) dt - \int_{t_0}^{t_1} q_\delta(x_1, t) dt}_{\text{no. of cars entering/leaving } [x_0, x_1]}. \quad (8.1.36)$$

In the limit of infinitely many cars this becomes exact and u, v, q become (piecewise) smooth functions: this justifies the transition to a differential (PDE) model.



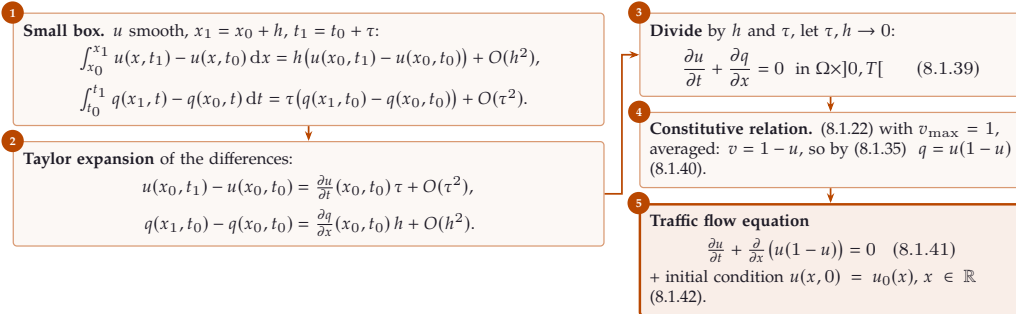
23 cars enter, 15 leave

Trajectories of Exp. 8.1.26 through a space-time box.

(8.1.37) The continuum limit

For $N \rightarrow \infty$ and $\delta \sim N^{-1} \rightarrow 0$ (drop the subscript δ) the balance law (8.1.36) remains valid and even becomes *exact*; u, v, q become (piecewise) smooth:

$$\int_{x_0}^{x_1} u(x, t_1) dx - \int_{x_0}^{x_1} u(x, t_0) dx = \int_{t_0}^{t_1} q(x_0, t) dt - \int_{t_0}^{t_1} q(x_1, t) dt. \quad (8.1.38)$$



8.1.3 Inviscid gas flow

Many particles ($\approx 10^{20}$ – 10^{30} molecules): the continuum limit is far better founded than for traffic. Frictionless (= **inviscid**) gas flow in an (infinitely) long pipe, $\Omega = \mathbb{R}$. Assumption: variation of gas density negligible (“near incompressibility”); motion of the fluid driven by inertia $\leftrightarrow$ conservation of linear momentum. Conservation of mass and energy are neglected (and violated).

- Unknown: $u = u(x, t)$ = momentum density $\sim$ local velocity $v = v(x, t)$ of the fluid.
- Flux of linear momentum $f \sim v \cdot u$, after scaling $f(u) = \frac{1}{2}u \cdot u$ (“momentum u advected by velocity u ”).

(8.1.43) Balance law: for all control volumes $V :=]x_0, x_1[\subset \Omega$, $0 < t_0 < t_1 < T$

$$\underbrace{\int_{x_0}^{x_1} u(x, t_1) - u(x, t_0) \, dx}_{\text{change of momentum in } V} + \underbrace{\int_{t_0}^{t_1} \frac{1}{2} u^2(x_1, t) - \frac{1}{2} u^2(x_0, t) \, dt}_{\text{outflow of momentum}} = 0. \quad (8.1.44)$$

(8.1.45) The Burgers equation

1

Small box. u smooth, $x_1 = x_0 + h$, $t_1 = t_0 + \tau$:

$$\int_{x_0}^{x_1} u(x, t_1) - u(x, t_0) dx = h(u(x_0, t_1) - u(x_0, t_0)) + O(h^2),$$

$$\int_{t_0}^{t_1} \frac{1}{2} u^2(x_1, t) - \frac{1}{2} u^2(x_0, t) dt = \tau(\frac{1}{2} u^2(x_1, t_0) - \frac{1}{2} u^2(x_0, t_0)) + O(\tau^2).$$



2

Taylor expansion:

$$u(x_0, t_1) - u(x_0, t_0) = \frac{\partial u}{\partial t}(x_0, t_0)\tau + O(\tau^2),$$

$$\frac{1}{2} u^2(x_1, t_0) - \frac{1}{2} u^2(x_0, t_0) = \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right)(x_0, t_0)h + O(h^2).$$



3

Divide by h and τ , let $\tau, h \rightarrow 0$: the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) = 0 \quad \text{in } \Omega \times]0, T[, \quad (8.1.46)$$

a one-dimensional scalar conservation law (without sources).

Remark 8.1.47: Euler equations

The gas model above ignores conservation of mass and of energy. A refined model of inviscid gas flow in an infinite pipe [15]:

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{pmatrix} = 0 \text{ in } \mathbb{R} \times]0, T[, \quad (8.1.48)$$

$u(x, 0) = u_0, \rho(x, 0) = \rho_0, E(x, 0) = E_0$, + state equation, e.g. ideal gas $p = (\gamma - 1)(E - \frac{1}{2}\rho u^2)$.

- $\rho \hat{=}$ fluid density, $[\rho] = \text{kg m}^{-3}$
- $u \hat{=}$ fluid velocity, $[u] = \text{m s}^{-1}$
- $p \hat{=}$ fluid pressure, $[p] = \text{N}$
- $E \hat{=}$ total energy density, $[E] = \text{J m}^{-3}$
- first equation $\leftrightarrow$ conservation of mass
- second equation $\leftrightarrow$ conservation of momentum
- third equation $\leftrightarrow$ conservation of energy

A **non-linear system** of conservation laws: not even existence and uniqueness for general initial values is established. This course: **scalar** conservation laws.

Review questions 8.1.49



1. Consider the Cauchy problem (8.1.4) for linear advection for $d = 2$ and $\mathbf{v}(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$. Write down the solution $u = u(x, t)$ in terms of the initial data $u_0 = u_0(x)$.
2. Show that your solution satisfies $\frac{\partial u}{\partial t} + \operatorname{div}_x(\mathbf{v}u) = 0$ in the sense of classical calculus, if u_0 is continuously differentiable.
3. In an x - t diagram sketch the trajectory of a car starting at $t = 0$, $x = 0$ and moving with constant acceleration to the right.
4. Which traffic flow conservation law arises when (8.1.22) is replaced with $v_{\text{opt}}(\Delta x) = v_{\text{max}} \cos\left(\frac{\pi}{2} \frac{\Delta_0}{\Delta x}\right)$?

8.2 *Scalar conservation laws in 1D*

8.2.1 Integral and differential form

Notation: $\Omega \subset \mathbb{R}^d$ fixed spatial domain ($\Omega = \mathbb{R}^d$: Cauchy problem), computational domain: the space-time cylinder $\tilde{\Omega} := \Omega \times]0, T[$, $U \subset \mathbb{R}^m$ phase (state) space; a vector $\in U$ is a *state*. Our focus: the scalar case $m = 1$.

Conservation law for a transient state distribution $u : \tilde{\Omega} \rightarrow U$

$$\underbrace{\frac{d}{dt} \int_V u \, dx}_{\text{change of amount}} + \underbrace{\int_{\partial V} \mathbf{f}(u, x) \cdot \mathbf{n} \, dS(x)}_{\text{inflow / outflow}} = \underbrace{\int_V s(u, x, t) \, dx}_{\text{production term}} \quad \forall \text{ "control volumes" } V \subset \Omega. \quad (8.2.1)$$

Flux function $\mathbf{f} : U \times \Omega \rightarrow \mathbb{R}^d$, source function $s : U \times \Omega \times]0, T[\rightarrow \mathbb{R}$ (here usually $s = 0$).

Burgers (8.1.46): $f(u) = \frac{1}{2}u^2$, $s = 0$; traffic flow (8.1.41): $f(u) = u(1 - u)$, $s = 0$; linear advection (8.1.2): $\mathbf{f}(u, x) = \mathbf{v}(x, t)u$, $s = f(x, t)$.

(8.2.1) has the same structure as the conservation of energy law for heat conduction:

$$\underbrace{\frac{d}{dt} \int_V \rho u \, dx}_{\text{energy stored in } V} + \underbrace{\int_{\partial V} \mathbf{j} \cdot \mathbf{n} \, dS}_{\text{power flux through } \partial V} = \underbrace{\int_V f \, dx}_{\text{heat generation in } V} \quad \text{for all control volumes } V, \quad (6.1.3)$$

with the heat flux given by Fourier's law $\mathbf{j}(x) = -\kappa(x) \mathbf{grad} u(x)$ (2.6.5), or its extended version (7.1.5): a *linear* function of derivatives of u .

Conversely, for f in (8.2.1) we assume

f depends only on the local state u , not on derivatives of u : $f(u, x) = f(u(x), x)$.

On the other hand

we go far beyond Fourier's law: f will, in general, be a *non-linear* function of u !

Remark 8.2.2 (Diffusive flux). A flux of the form of Fourier's law, $f(u) = -\kappa(x) \mathbf{grad} u$, is called a diffusive flux.

Integrate (8.2.1) over $[t_0, t_1] \subset [0, T]$ (space-time box, cf. (8.1.44)):

$$\int_V u(x, t_1) dx - \int_V u(x, t_0) dx + \int_{t_0}^{t_1} \int_{\partial V} \mathbf{f}(u, x) \cdot \mathbf{n} dS(x) dt = \int_{t_0}^{t_1} \int_V s(u, x, t) dx dt \quad (8.2.3)$$

for all $V \subset \Omega$, $0 < t_0 < t_1 < T$, $\mathbf{n} \hat{=}$ exterior unit normal at ∂V .

Gauss' theorem (Thm. 2.5.7) $\Rightarrow$ (local) **differential form** of (8.2.1), div acting on x only:

$$\frac{\partial}{\partial t} u + \text{div}_x \mathbf{f}(u, x) = s(u, x, t) \quad \text{in } \tilde{\Omega}, \quad u(x, 0) = u_0(x), \quad x \in \Omega. \quad (8.2.4)$$

Special case $d = 1$, a “density” $u : \tilde{\Omega} \rightarrow \mathbb{R}$:

$$\frac{\partial u}{\partial t}(x, t) + \frac{\partial}{\partial x} (f(u(x, t), x)) = s(u(x, t), x, t) \quad \text{in }]\alpha, \beta[\times]0, T[. \quad (8.2.5)$$

Remark 8.2.6. Boundary values: tricky, f -dependent (transport direction depends on the unknown u , cf. Rem. 8.1.13) $\Rightarrow$ only Cauchy problems in this chapter.

8.2.2 Characteristics

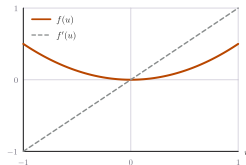
Cauchy problem ($\Omega = \mathbb{R}$) for the one-dimensional scalar conservation law (8.2.5):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad \text{in } \mathbb{R} \times]0, T[, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}. \quad (8.2.7)$$

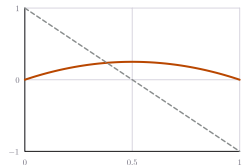
Chain rule: $(8.2.7) \Leftrightarrow \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial x} = 0$, cf. (8.1.10): **$f'(u)$ is a u -dependent velocity of transport.**

Assumption 8.2.8 (Monotonicity of f')

The flux function $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth ($f \in C^2$) and convex or concave [53, Def. 5.5.2] ($\Rightarrow f'$ increasing resp. decreasing).



Burgers: $f(u) = \frac{1}{2}u^2$ (convex)



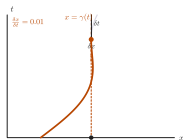
traffic: $f(u) = u(1-u)$ (concave)

Definition 8.2.9 (Characteristic curve)

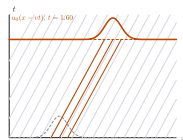
A curve $\Gamma := (\gamma(\tau), \tau) : [0, T] \rightarrow \mathbb{R} \times]0, T[$ in the (x, t) -plane is a characteristic curve for (8.2.7), if

$$\frac{d}{d\tau} \gamma(\tau) = f'(u(\gamma(\tau), \tau)), \quad 0 \leq \tau \leq T, \quad (8.2.10)$$

where u is a continuously differentiable solution of (8.2.7).



u curve in the x - t plane – a point moving on the real axis



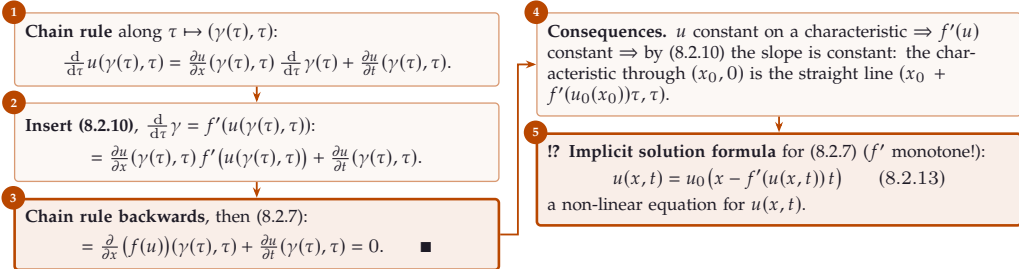
advection: profiles slide along $\gamma(\tau) = v\tau + c$

Left: $x = \gamma(t) \leftrightarrow$ movement of a point on the real axis with speed $\frac{d}{d\tau} \gamma$.

Example 8.2.11. Constant linear advection $f(u) = vu$: characteristics $\gamma(\tau) = v\tau + c$; the solution $u_0(x - vt)$ is meaningful for any u_0 . Characteristics generalize streamlines (Sect. 7.1.1).

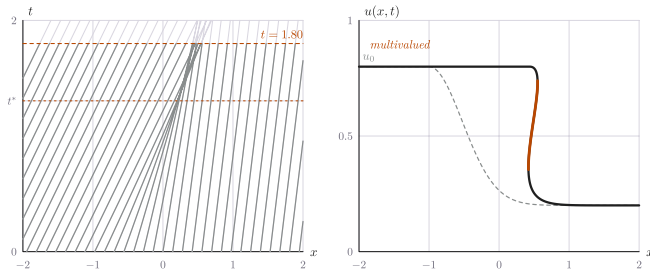
Lemma 8.2.12: constancy along characteristics

Lemma 8.2.12 (Classical solutions and characteristic curves). Smooth solutions of (8.2.7) are constant along characteristic curves.



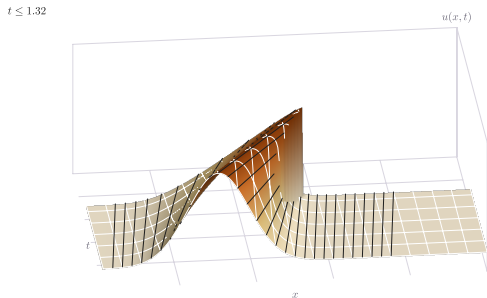
(8.2.14) Breakdown of (8.2.13): Burgers

The key problem of (8.2.13): it may have **multiple solutions**. Burgers' equation (8.1.46), $f(u) = \frac{1}{2}u^2$ smooth and strictly convex, $f'(u) = u$ increasing, u_0 smooth and *decreasing*:



Characteristics $x_0 + u_0(x_0)t$ and the profile by (8.2.13): from $t^* = 1/\max(-u'_0)$ on they intersect and the wave breaks, a “multivalued solution”.

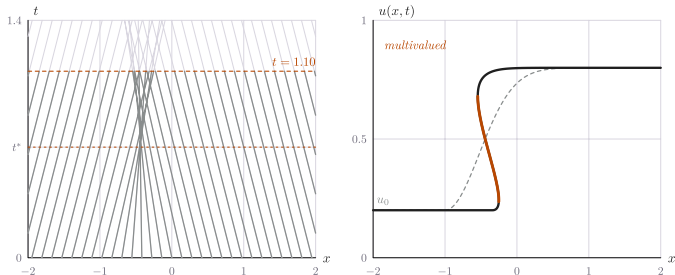
The breaking wave in 3D



The entropy solution $u(x, t)$ of Burgers' equation as a surface over the x - t plane, built up in time. Black: characteristics $x_0 + u_0(x_0)t$ drawn *on* the surface at height $u_0(x_0)$: by Lemma 8.2.12 they are level lines. Where they collide, the surface cannot fold over and tears into a shock.

(8.2.14) Breakdown of (8.2.13): traffic flow

Traffic flow equation (8.1.41), $f(u) = u(1 - u)$ smooth and strictly concave, $f'(u) = 1 - 2u$ decreasing, u_0 smooth and *increasing*:



Consequence: breakdown of classical solutions & Ex. 8.2.11 $\Rightarrow$ a new concept of solution of (8.2.7).

Remark 8.2.15: meaning of characteristics

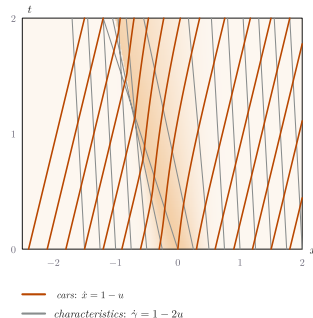
For traffic flow, $f(u) = u(1 - u)$:

- characteristics: $\dot{\gamma}(t) = -2u(\gamma(t), t) + 1$,
- car trajectories: $\dot{x}(t) = 1 - u(x(t), t)$.

Characteristics do *not* give the paths of cars: cars always drive to the right, while characteristics may be slanted to the left!

Yet, by Lemma 8.2.12, for a smooth solution the characteristic through (x^*, t^*) is the locus of space-time points on which $u(x^*, t^*)$ depends ($t < t^*$) or which it influences ($t > t^*$).

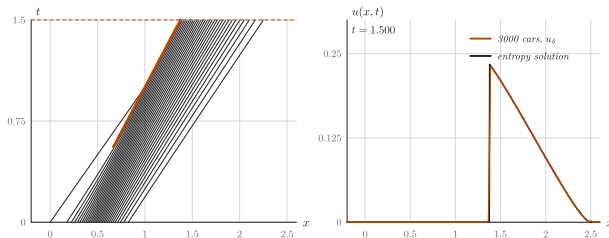
For a scalar conservation law **information “flows” along characteristic curves.**



A density bump on $u \approx 0.55$: cars (orange) pass through it to the right; the bump itself travels left along the characteristics.

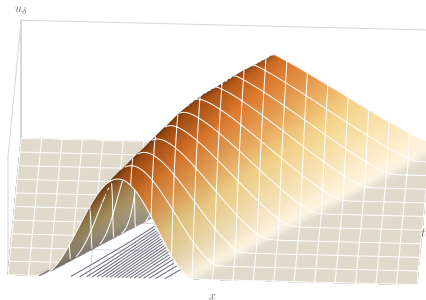
Ex. 8.2.16: smooth initial density, 3000 cars

Particle model (8.1.21), (8.1.22) with $N = 3000$ cars placed from the smooth density $u_0(x) = 2 \sin^2(\pi x)$ on $[0, 1]$: $x_i(0) = \Phi^{-1}(\frac{i-1}{N-1})$, $\Phi(\xi) = \int_0^\xi u_0(x) dx$ (normalized peak density 0.25).



Every 100th car; density (8.1.30) against the entropy solution (Sect. 8.2.6). A discontinuity crops up from smooth data: an intrinsic feature of the model, reflecting physical reality.

3000 cars, seen as a surface



The car density $u_\delta(x, t)$ of Ex. 8.2.16 (3000 cars, triangular window) as a surface; on the floor every 150th trajectory. The front of the bump steepens into a shock while the cars keep driving to the right.

8.2.3 Weak solutions

Discontinuous solutions are not classical; cf. the elastic string with a kink (Ex. 1.3.37). The integral form (8.2.3) points the way.

"Space-time Gaussian theorem"

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \quad (8.2.17)$$

$\Downarrow$

$$\operatorname{div}_{(x,t)} \begin{pmatrix} f(u) \\ u \end{pmatrix} = 0 \quad \text{in } \tilde{\Omega}, \quad (8.2.18)$$

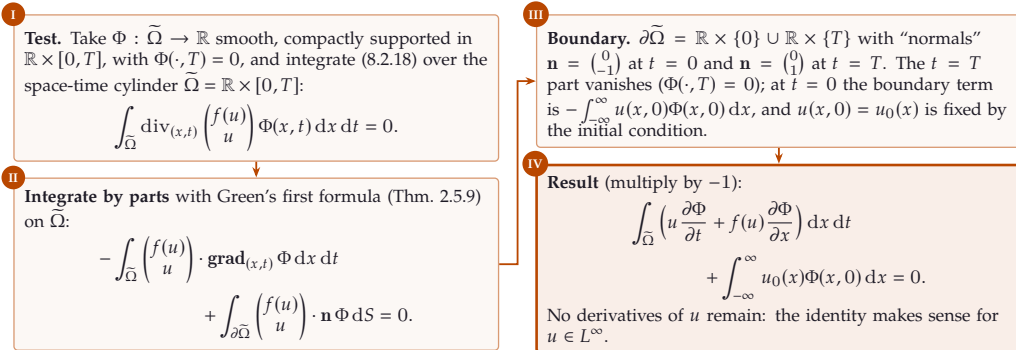
$$\forall \text{ space-time control volumes } \tilde{V} \subset \tilde{\Omega}: \\ \int_{\partial \tilde{V}} \begin{pmatrix} f(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_t \end{pmatrix} dS = 0.$$

For a space-time rectangle $\tilde{V} =]x_0, x_1[\times]t_0, t_1[$:

$$\begin{aligned} & \int_{x_0}^{x_1} u(x, t_1) dx - \int_{x_0}^{x_1} u(x, t_0) dx \\ &= \int_{t_0}^{t_1} f(u(x_0, t)) dt - \int_{t_0}^{t_1} f(u(x_1, t)) dt. \end{aligned} \quad (8.2.19)$$

Problem: a discontinuity of u on an edge of the rectangle. Idea as in Sect. 2.9: **I** test with a smooth function, **II** integrate by parts in space & time, **III** use the initial conditions.

Derivation of the weak form (steps I–III)



Definition 8.2.20 (Weak solution of Cauchy problem for scalar conservation law)

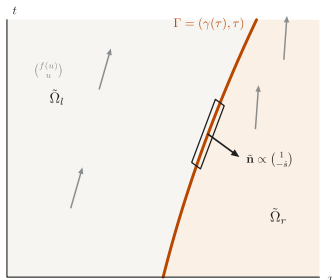
For $u_0 \in L^\infty(\mathbb{R})$, $u : \mathbb{R} \times]0, T[\rightarrow \mathbb{R}$ is a weak solution of the Cauchy problem (8.2.7), if $u \in L^\infty(\mathbb{R} \times]0, T[)$ and

$$\int_{-\infty}^{\infty} \int_0^T \left(u \frac{\partial \Phi}{\partial t} + f(u) \frac{\partial \Phi}{\partial x} \right) dt dx + \int_{-\infty}^{\infty} u_0(x) \Phi(x, 0) dx = 0$$

for all $\Phi \in C_0^\infty(\mathbb{R} \times [0, T[)$, $\Phi(\cdot, T) = 0$.

Remark 8.2.21 (Properties of weak solutions). Reversing the integration by parts: u weak solution & $u \in C^1 \iff$ classical solution. A weak solution $u \in L_{\text{loc}}^\infty$ satisfies the integral form (8.2.19) for “almost all” $x_0 < x_1, 0 < t_0 < t_1 < T$.

8.2.4 Jump conditions: the pillbox argument



A slender pillbox along the curve of discontinuity: unless the normal components match, a net flux through its boundary results.

- 1 **Normal continuity.** For a piecewise smooth $\mathbf{j}$ on $\Omega \subset \mathbb{R}^2$: $\operatorname{div} \mathbf{j} = 0 \iff \int_{\partial V} \mathbf{j} \cdot \mathbf{n} \, dS = 0 \, \forall V$. Necessary: normal components are continuous across discontinuities. Apply to $\mathbf{j} = \begin{pmatrix} f(u) \\ u \end{pmatrix}$, divergence free by (8.2.18).
- 2 **Normal of the curve** $\Gamma := \tau \mapsto (\gamma(\tau), \tau)$: the tangent is $\begin{pmatrix} \dot{\gamma} \\ 1 \end{pmatrix}$, so $\tilde{\mathbf{n}} = \frac{1}{\sqrt{1+|\dot{\gamma}|^2}} \begin{pmatrix} 1 \\ -\dot{\gamma} \end{pmatrix}$, $\dot{\gamma} := \frac{d\gamma}{d\tau}$ “speed of the curve”.
- 3 **Rankine–Hugoniot (jump) condition.** With $\llbracket \cdot \rrbracket$ the jump across Γ “from left to right”, e.g. $\llbracket u \rrbracket = u_l - u_r$:

$$\begin{pmatrix} 1 \\ -\frac{d\gamma}{d\tau} \end{pmatrix} \cdot \begin{pmatrix} \llbracket f(u) \rrbracket \\ \llbracket u \rrbracket \end{pmatrix} = 0 \quad (8.2.22)$$
 shorthand: $\dot{\gamma} (u_l - u_r) = f_l - f_r$ (8.2.23), $\dot{\gamma}$ = propagation speed of the discontinuity.

Constant states (8.2.24) and the Riemann problem

Simplest situation compliant with (8.2.22):
constant states left and right of the line of
discontinuity,

$$u(x, t) = \begin{cases} u_l \in \mathbb{R}, & x < \dot{s}t, \\ u_r \in \mathbb{R}, & x > \dot{s}t, \end{cases} \quad (8.2.25)$$

with constant speed, by (8.2.23) (for $u_l \neq u_r$)

$$\dot{s} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Definition 8.2.26 (Riemann problem)

The Cauchy problem (8.2.7) with piecewise
constant initial data

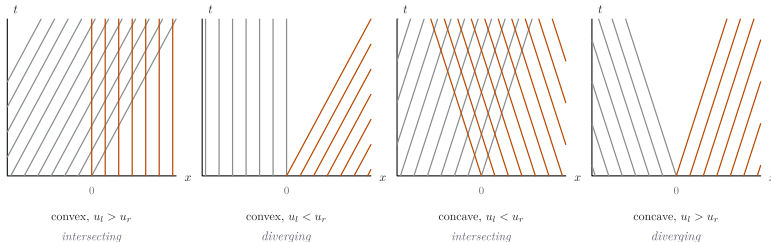
$$u_0(x) = \begin{cases} u_l \in \mathbb{R}, & x < 0, \\ u_r \in \mathbb{R}, & x > 0 \end{cases}$$

is the *Riemann problem* for (8.2.7).

Locally constant states are particularly easy: the
Riemann problem is the building block of
everything that follows, also numerically
(Godunov, Lecture II).

Riemann problem: characteristic patterns

For f smooth & convex, f' non-decreasing; for f smooth & concave, f' non-increasing (cf. Sect. 8.2.2).
Characteristics $x = x_0 + f'(u_0(x_0))t$ of the Riemann data:



Intersecting characteristics: a discontinuity is needed at once. *Diverging* characteristics: a wedge without information, which the solution must fill.

Definition 8.2.27 (Shock)

If Γ is a smooth curve in the (x, t) -plane and u a weak solution of (8.2.7), a discontinuity of u across Γ is called a *shock*.

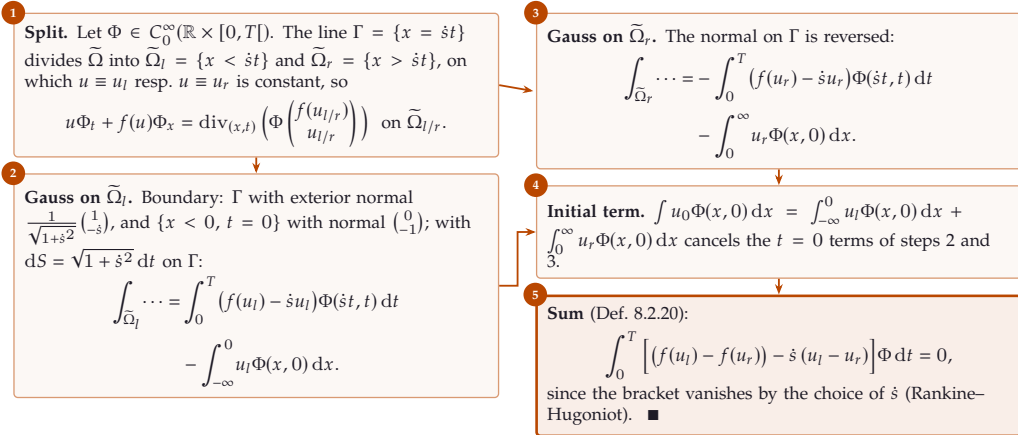
By §8.2.24 the shock speed is given by the Rankine–Hugoniot jump conditions: for $(x_0, t_0) \in \Gamma$

$$\dot{s} = \frac{f(u_l) - f(u_r)}{u_l - u_r}, \quad u_l := \lim_{\epsilon \rightarrow 0} u(x_0 - \epsilon, t_0), \quad u_r := \lim_{\epsilon \rightarrow 0} u(x_0 + \epsilon, t_0). \quad (8.2.28)$$

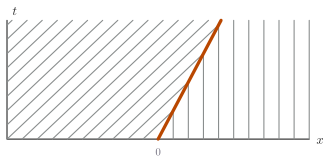
Lemma 8.2.29 (Shock solution of Riemann problem)

For any two states $u_l, u_r \in \mathbb{R}$ the piecewise constant function $u(x, t) := u_l$ for $x < \dot{s}t$, $u(x, t) := u_r$ for $x > \dot{s}t$ ($x \in \mathbb{R}$, $0 < t < T$), with $\dot{s} := \frac{f(u_l) - f(u_r)}{u_l - u_r}$, is a weak solution (Def. 8.2.20) of the related Riemann problem for (8.2.7).

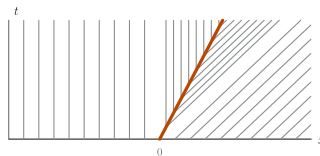
Proof of Lemma 8.2.29



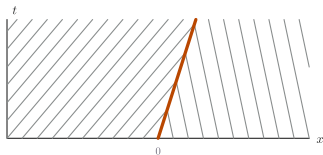
Shocks: dependence on u_l, u_r



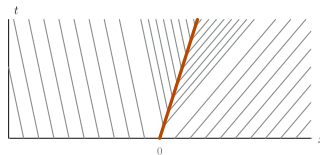
Burgers, $u_l = 1 > u_r = 0$ *characteristics impinge*



Burgers, $u_l = 0 < u_r = 1$ *characteristics emanate*



traffic, $u_l = 0.1 < u_r = 0.6$ *characteristics impinge*

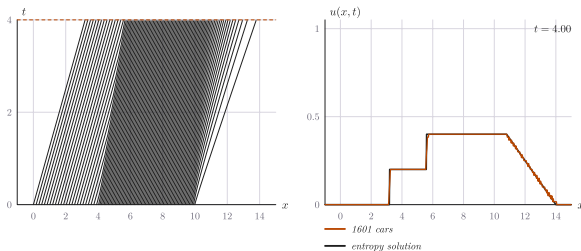


traffic, $u_l = 0.6 > u_r = 0.1$ *characteristics emanate*

Burgers: for $u_l > u_r$ characteristics impinge on the shock, for $u_l < u_r$ they emanate (*expansion shock*); traffic flow: the mirror image.

Ex. 8.2.30: shock patterns in traffic flow

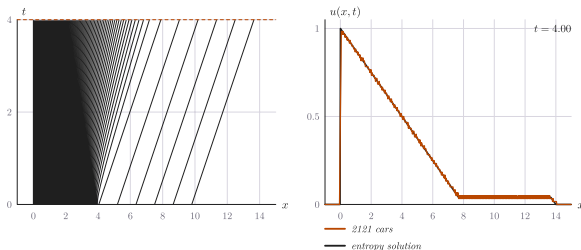
Particle model as in Exp. 8.1.31 with $x_0 = [(0 : 0.01 : 4), (4.005 : 0.005 : 10)]$ (1601 cars), $\Delta_0 = 0.002$: a column of fast going cars (density 0.2) approaches a zone of dense traffic (density 0.4).



Observation: shocks present in the initial conditions persist and travel with the speed predicted by the jump conditions (8.2.23): $\dot{s} = \frac{f(0.2) - f(0.4)}{0.2 - 0.4} = 1 - 0.2 - 0.4 = 0.4$.

Ex. 8.2.31: fan patterns in traffic flow

Initial car distribution $x_0 = [(0 : 0.002 : 4), (4.05 : 0.05 : 10)]$ (2121 cars), $\Delta_0 = 0.002$: the front end of a traffic jam (density 1) next to sparse traffic (density 0.04).

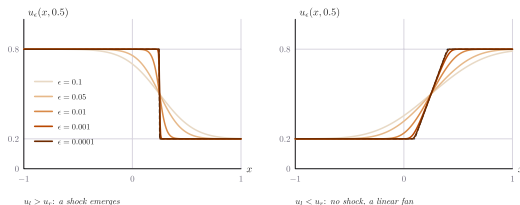


Observation: the jump disappears, replaced by a zone of *linearly* decreasing density whose edges move with constant speeds in opposite directions. **No shock solution!**

Ex. 8.2.32: vanishing viscosity (Burgers)

No “inviscid” fluid exists: conservation laws are limits $\epsilon = 0$ of transport-diffusion problems, which have unique smooth solutions; $\epsilon \rightarrow 0$ tells the physically meaningful one (**vanishing viscosity method**).

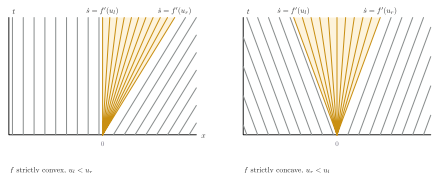
Viscous Burgers equation:
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) = \underbrace{\epsilon \frac{\partial^2 u}{\partial x^2}}_{\text{dissipative (viscous) term}}. \quad (8.2.33)$$



Cole–Hopf [23]: $u_\epsilon = w(x - \dot{s}t)$, $w(\xi) = u_r + \frac{1}{2}(u_l - u_r)(1 - \tanh \frac{\xi(u_l - u_r)}{4\epsilon})$, $\dot{s} = \frac{u_l + u_r}{2}$.

(8.2.34) Similarity solution

- 1 **Scaling invariance.** (8.2.17) is homogeneous in the spatial/temporal derivatives: with $u_\lambda(x, t) := u(\lambda x, \lambda t)$, $\lambda > 0$,
 $\frac{\partial u_\lambda}{\partial t} + \frac{\partial}{\partial x} f(u_\lambda) = 0$; Riemann data satisfy $u_0(\lambda x) = u_0(x)$.
- 2 **Ansatz** $u(x, t) = \psi(x/t)$, constant on lines through $(0, 0)$. Insert:
 $-\frac{x}{t^2} \psi'(\frac{x}{t}) + f'(\psi(\frac{x}{t})) \frac{1}{t} \psi'(\frac{x}{t}) = 0$.
- 3 **Hence** $f'(\psi(w))\psi'(w) = w \psi'(w)$ for $w = x/t$:
 $\psi' \equiv 0 \quad \vee \quad f'(\psi(w)) = w \iff \psi(w) = (f')^{-1}(w)$,
 using that f' is strictly monotone.



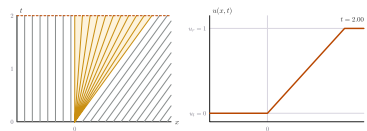
Similarity solutions for Riemann data: a fan of characteristics through the origin between the speeds $f'(u_l)$ and $f'(u_r)$.

Lemma 8.2.35 (Rarefaction solution of Riemann problem)

If $f \in C^2(\mathbb{R})$ is strictly convex and $u_l < u_r$, or strictly concave and $u_r < u_l$, $g := (f')^{-1}$:

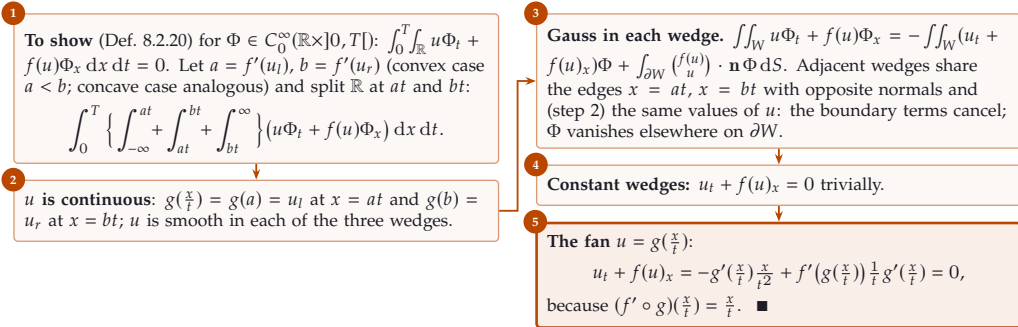
$$u(x, t) := \begin{cases} u_l, & x < a t, \\ g(\frac{x}{t}), & a < \frac{x}{t} < b, \\ u_r, & x > b t, \end{cases}$$

$a := \min\{f'(u_l), f'(u_r)\}$, $b := \max\{f'(u_l), f'(u_r)\}$, is a weak solution of the Riemann problem: a **rarefaction wave**, a continuous solution!

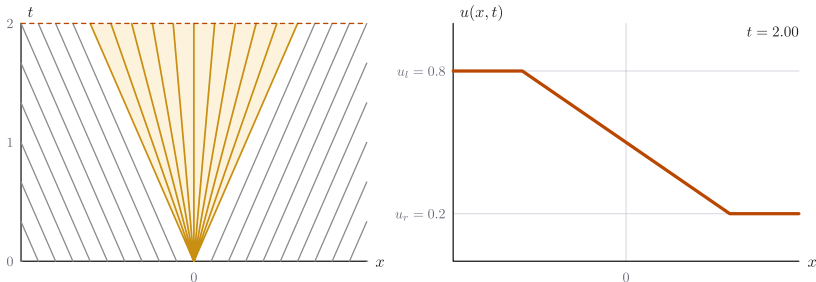


Burgers, $u_l = 0 < u_r = 1$: the fan $u = x/t$ between $x = 0$ and $x = t$.

Proof of Lemma 8.2.35

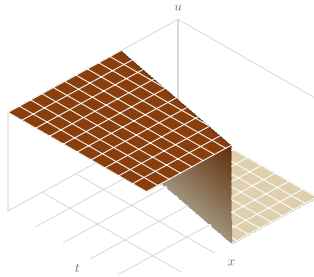


Rarefaction wave: traffic flow

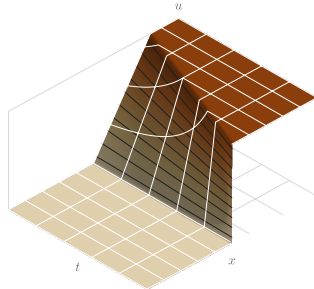


Traffic flow flux $f(u) = u(1 - u)$, $u_l = 0.8 > u_r = 0.2$: $g(w) = \frac{1}{2}(1 - w)$; the fan spans the speeds $f'(0.8) = -0.6$ to $f'(0.2) = 0.6$, exactly the linearly decreasing density of the front of a jam in Ex. 8.2.31.

Shock and rarefaction as surfaces



shock ($u_l = 1 > u_r = 0$)

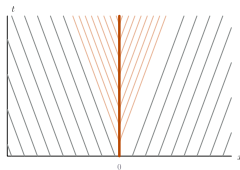


rarefaction ($u_l = 0 < u_r = 1$): $u = x/t$

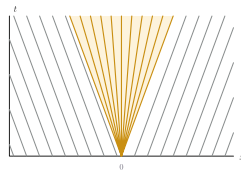
Riemann problems for Burgers. Left: the shock of Lemma 8.2.29, a step moving with $\dot{s} = \frac{1}{2}$. Right: the rarefaction of Lemma 8.2.35 is a *ruled* surface: each ray $x = ct$ through the origin carries the single value $u = c$ (black).

8.2.6 Entropy condition

Weak solutions need not be unique: for f' decreasing (traffic flow) and $u_l > u_r$ both a shock and a rarefaction wave are weak solutions.



shock: characteristics emanate
 $\Rightarrow$ violates Def. 8.2.36



rarefaction wave: the entropy solution

How to select the “physically meaningful” = **admissible** solution?

- ① Comparison with microscopic models, see Ex. 8.2.31 for traffic flow.
- ② Vanishing viscosity (Ex. 8.2.32): add an “ ϵ -amount” of diffusion, study $\epsilon \rightarrow 0$.

Desirable: simple selection criteria (**entropy conditions**).

Definition 8.2.36 (Lax entropy condition)

u = weak solution of (8.2.7), piecewise classical in a neighborhood of a C^2 -curve $\Gamma := (\gamma(\tau), \tau)$, $0 \leq \tau \leq T$, discontinuous across Γ . u satisfies the Lax entropy condition in $(x_0, t_0) \in \Gamma : \iff$

$$f'(u_l) > \dot{s} := \frac{f(u_l) - f(u_r)}{u_l - u_r} > f'(u_r).$$

⇕ Characteristics must not emanate from a shock: no “generation of information”; expansion shocks are not allowed.

Such a shock = **physical shock**.

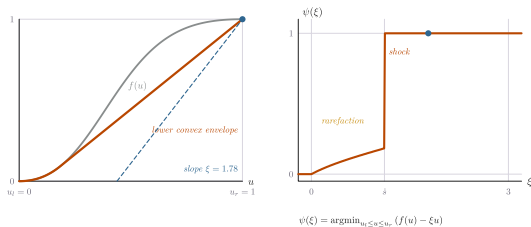
Entropy solution

Physically meaningful weak solution = *entropy solution*; for Lipschitz flux they exist and are unique [23].

Rem. 8.2.37: the general Riemann solution

For arbitrary $f \in C^1(\mathbb{R})$ there is a formula for the entropy solution of the Riemann problem [41]:

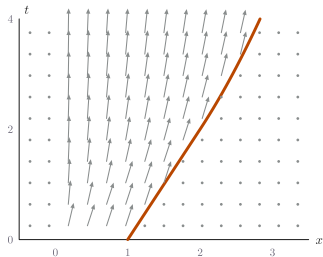
$$u(x, t) = \psi(x/t), \quad \psi(\xi) := \begin{cases} \operatorname{argmin}_{u_l \leq u \leq u_r} (f(u) - \xi u), & \text{if } u_l < u_r, \\ \operatorname{argmax}_{u_r \leq u \leq u_l} (f(u) - \xi u), & \text{if } u_l \geq u_r. \end{cases} \quad (8.2.38)$$



$\psi(\xi)$: where a line of slope ξ supports the lower convex envelope of f ; here $f(u) = \frac{u^2}{u^2 + (1-u)^2/2}$.

Ex. 8.2.39: entropy solution, Burgers

Analytic solution for Burgers equation (8.1.46) with $u_0 = 1$ on $[0, 1]$, 0 elsewhere [23, Sect. 3.4, Ex. 3]: a rarefaction $u = x/t$ at $x = 0$, a shock at $x = 1$ with speed $\frac{1}{2}$ until $t = 2$; then the fan catches up with the shock, $\dot{s} = \frac{1}{2}u_l = \frac{\gamma}{2t} \Rightarrow \gamma(t) = \sqrt{2t}$.

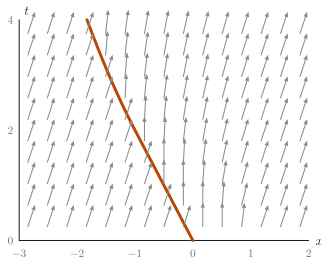


the field $\left(f_u^{(u)}\right)$ is tangential to the shock
(normal continuity across Γ)

Last frame: the field $\left(f_u^{(u)}\right)$ in the x - t plane, tangential to the shock (normal continuity).

Ex. 8.2.40: entropy solution, traffic flow

Traffic flow equation (8.1.41) with $u_0 = 1$ on $[0, 1]$, 0.5 elsewhere: a fan $u(x, t) = \frac{1}{2}\left(1 - \frac{x-1}{t}\right)$ at $x = 1$, a shock at $x = 0$ with speed $-\frac{1}{2}$, which meets the fan at $t = 2$ and then follows $\gamma(t) = 1 - \sqrt{2t}$.



the field $(f_u^{(u)})$ is tangential to the shock
(normal continuity across Γ)

Characteristics end on the shock from both sides (Def. 8.2.36); last frame: normal continuity.

8.2.7 Properties of entropy solutions

$u \in L^\infty(\mathbb{R} \times]0, T[)$ weak entropy solution of (8.2.7),
 $f \in C^1(\mathbb{R})$ (not necessarily convex/concave); $\bar{u} \triangleq$ entropy
solution for initial data $\bar{u}_0 \in L^\infty(\mathbb{R})$.

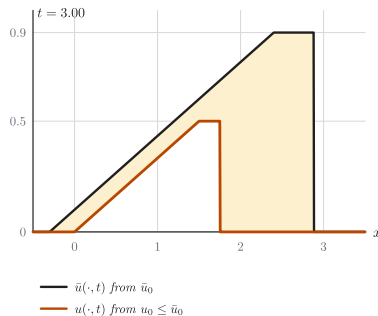
Theorem 8.2.41 (Comparison principle)

$$u_0 \leq \bar{u}_0 \text{ a.e. on } \mathbb{R} \Rightarrow u \leq \bar{u} \text{ a.e. on } \mathbb{R} \times]0, T[.$$

Constant data give constant solutions: $u_0(x) \in [\alpha, \beta] \Rightarrow$
 $u(x, t) \in [\alpha, \beta]$ (guarantees $0 \leq u \leq 1$ for traffic flow),
and

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \quad \forall 0 \leq t \leq T, \quad (8.2.42)$$

L^∞ -stability: no blow-up can occur!



Theorem 8.2.43 (L^1 -contractivity of evolution for scalar conservation law)

$$\forall t \in]0, T[, R > 0 : \int_{|x| < R} |u(x, t)| dx \leq \int_{|x| < R + \dot{s}t} |u_0(x)| dx, \text{ with maximal speed of propagation} \\ \dot{s} := \max\{|f'(\xi)| : \inf_x u_0 \leq \xi \leq \sup_x u_0\} \quad (8.2.44)$$

Finite speed of propagation, as for the wave equation (Sect. 6.2.2):

- domain of dependence

$$D^-(\bar{x}, \bar{t}) := \{(x, t) : \bar{x} - \dot{s}t \leq x \leq \bar{x} + \dot{s}t\},$$

- domain of influence

$$D^+([a, b]) := \{(x, t) : a - \dot{s}t \leq x \leq b + \dot{s}t\}$$

No oscillations: the number of local extrema of $u(\cdot, t)$ decreases with time.

Corollary 8.2.45 (Domain of dependence)

The value of the entropy solution at $(\bar{x}, \bar{t}) \in \tilde{\Omega}$ depends only on the restriction of u_0 to $\{x : |x - \bar{x}| < \dot{s}\bar{t}\}$.

Review questions 8.2.46

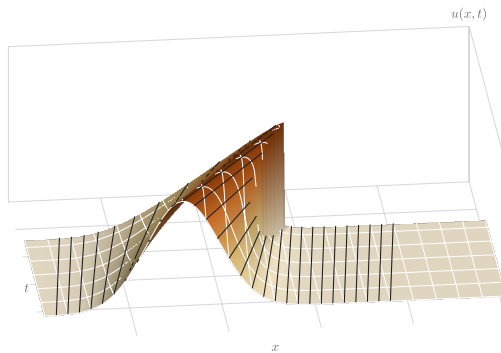


1. Write down the general form of a Cauchy problem for a 1D scalar conservation law (without source terms).
2. For flux functions (a) $f(u) = u^2$, (b) $f(u) = \sin(\pi u)$, (c) $f(u) = \cos(\pi u)$ and initial data $u_0(x) = 1$ for $-1 \leq x \leq 1$, $u_0(x) = 0$ elsewhere, sketch the family of characteristic curves (Def. 8.2.9) in an $x-t$ diagram.
3. Show that $u(x, t) = u_0(x - vt)$, $u_0 \in L^\infty(\mathbb{R})$, is a weak solution of $\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0$.
4. What is the Lax entropy condition and why is it important?
5. For $u_0(x) = 0$ for $x < 0$, $u_0(x) = 1$ for $x \geq 0$ give the entropy solutions of the Riemann problems for the flux functions (a) $f(u) = u^4$, (b) $f(u) = \log(1 + u)$, (c) $f(u) = 1 - e^u$, (d) $f(u) = \frac{1}{1+u}$.
6. Explain the notions of “domain of dependence” and “domain of influence” for Cauchy problems for 1D scalar conservation laws.
7. Which formula yields the maximal speed of propagation for a Cauchy problem for a 1D scalar conservation law?

- Conservation laws (8.2.1): change of amount + flux through the boundary = production. Examples: linear advection, traffic flow (from a particle model), Burgers (inviscid gas), Euler.
- Characteristics carry information; for non-linear flux they **cross**: classical solutions break down.
- **Weak solutions** (Def. 8.2.20) admit discontinuities; across them the Rankine–Hugoniot condition (8.2.23).
- Riemann problem: shocks (Lemma 8.2.29) and rarefaction waves (Lemma 8.2.35); weak solutions are not unique.
- The **Lax entropy condition** (Def. 8.2.36) selects the physical one; in general, (8.2.38).
- Entropy solutions: comparison principle, L^∞ -stability, L^1 -contractivity, finite speed of propagation, no new extrema.

Next lecture. Discretizations that respect conservation: finite volumes in conservation form, numerical fluxes, CFL.

$$t \leq 1.32$$



Thank you.

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