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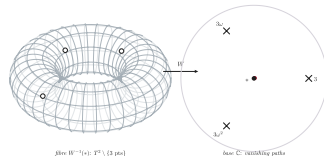


Homological Mirror Symmetry II

Landau–Ginzburg models, hypersurfaces and singular fibres

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Lecture I. For Calabi–Yau pairs, $D^\pi \mathcal{F}(X) \simeq D^b \text{Coh}(\check{X})$; on the elliptic curve triangles are theta functions; mirrors come from dual torus fibrations, and tropical geometry is their shadow.

Today. What happens when X is *not* Calabi–Yau, or not compact, or *singular*? Mirrors acquire a function W , sheaves become matrix factorizations, and the singular fibre needs a definition.

A-side

Fano X (e.g. $\mathbb{CP}^2$)

LG model (Y, W)

very affine hypersurface $H \subset (\mathbb{C}^*)^n$

singular fibre $f^{-1}(0)$

B-side

Landau–Ginzburg model $((\mathbb{C}^*)^n, W)$

matrix factorizations / $\text{Sing}(W^{-1}(0))$

toric boundary divisor $D \subset X$

? (this is where Jeffs’s thesis starts)

1

Landau–Ginzburg models and Lefschetz fibrations

The mirror of a Fano manifold is a function

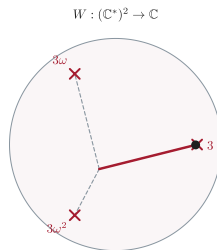
A Landau–Ginzburg model is (Y, W) with $W : Y \rightarrow \mathbb{C}$ holomorphic. A toric Fano X with polytope $\{\langle v_i, u \rangle \geq -\lambda_i\}$ has mirror (Givental, Hori–Vafa)

$$W = \sum_i T^{\lambda_i} z^{v_i} \quad \text{on } (\mathbb{C}^*)^n.$$

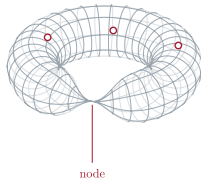
$\mathbb{CP}^2$

$W = x + y + \frac{T}{xy}$. Critical points: $x = y = w$
with $w^3 = T$, three of them, critical values $3w$.

Three critical points, and $\text{rk } H^*(\mathbb{CP}^2) = 3$.



$W^{-1}(b) : T^2 \setminus \{3 \text{ punctures}\}$



Moving to a critical value, a circle in the fibre (an elliptic curve minus three points) shrinks to a node.

Vanishing cycles and the Fukaya–Seidel category

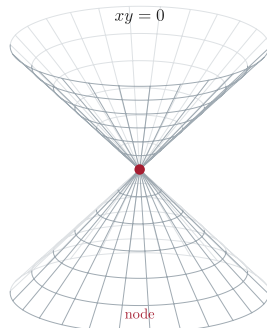
Near a critical point $W = z_1^2 + \cdots + z_n^2$. Along a path to the critical value a sphere $V \subset W^{-1}(b)$ shrinks to the point; it sweeps out the *Lefschetz thimble* Δ , a Lagrangian disk.

Fukaya–Seidel category $\mathcal{F}(Y, W)$

Objects: thimbles. $\text{Hom}(\Delta_i, \Delta_j) = CF(V_i, V_j)$ for $i < j$ and 0 for $i > j$: a *directed* category.

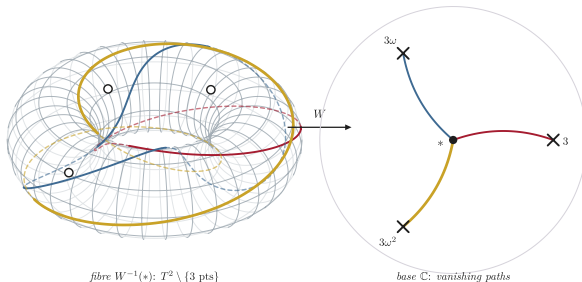
Seidel; Auroux–Katzarkov–Orlov

$D^\pi \mathcal{F}((\mathbb{C}^*)^2, W) \simeq D^b \text{Coh}(\mathbb{CP}^2)$: thimbles $\leftrightarrow \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$.



The local model $\{xy = t\}$: the vanishing cycle is the waist circle; at $t = 0$ the node.

Vanishing paths for the mirror of $\mathbb{CP}^2$



$W = x + y + \frac{1}{xy}$: one path from $*$ to each critical value; along it a circle V_i in the fibre $T^2 \setminus \{3 \text{ pts}\}$ shrinks to the node. Classes $(1, -1), (1, 2), (-2, -1)$, pairwise meeting in 3 points: $\text{Hom}(\Delta_i, \Delta_j)$ of rank 3, the collection $(\mathcal{O}, T(-1), \mathcal{O}(1))$; other bases of paths (Hurwitz moves) give its mutations, e.g. $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ [AKO].

Monodromy is a Dehn twist

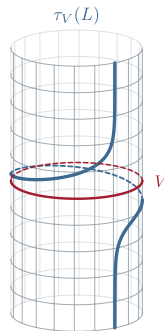
Picard–Lefschetz

The monodromy around a critical value is the Dehn twist τ_V about the vanishing cycle:
 $\tau_V(L) = L \pm \langle L, V \rangle V$ in homology.

Seidel's exact triangle

$\tau_V(L) \simeq \text{Cone}(HF(V, L) \otimes V \rightarrow L)$ in $\mathcal{F}$.

For later. The monodromy is a functor μ with a natural transformation $s : \mu \rightarrow \text{id}$ [Seidel].



2

Matrix factorizations and Knörrer periodicity

The B-side of an LG model: matrix factorizations

Matrix factorization of W

A pair of maps $E_1 \xrightarrow{\phi_1} E_0 \xrightarrow{\phi_0} E_1$ of vector bundles with $\phi_0\phi_1 = W \cdot \text{id}$ and $\phi_1\phi_0 = W \cdot \text{id}$. They form a 2-periodic dg category $\text{MF}(W)$.

Orlov 2004

$$\text{MF}(W) \simeq \text{Sing}(W^{-1}(0)) := D^b\text{Coh}(W^{-1}(0))/\text{Perf}.$$

On $W^{-1}(0)$ a matrix factorization is a 2-periodic resolution that never stops; perfect complexes are killed, what remains is the singularity.

$$\cdots \mathcal{O} \xrightarrow{x} \mathcal{O} \xrightarrow{y} \mathcal{O} \xrightarrow{x} \mathcal{O} \xrightarrow{y} \mathcal{O} \cdots$$

$$x \cdot y = W = y \cdot x$$

$W = xy$: the factorization (x, y) resolves $\mathcal{O}_{\{x=0\}}$ on the node $\{xy = 0\}$, 2-periodically and without end.

Knörrer 1987; derived: Orlov 2006

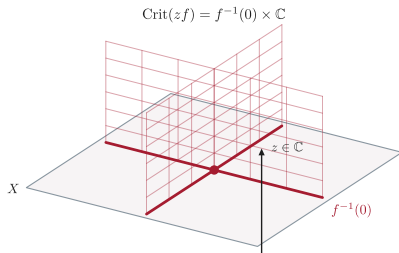
For X smooth and $f : X \rightarrow \mathbb{C}$,

$$D^b \text{Coh}(f^{-1}(0)) \simeq \text{MF}(X \times \mathbb{C}, zf),$$

where z is the coordinate on $\mathbb{C}$.

$\text{Crit}(zf) \supset f^{-1}(0) \times \mathbb{C}$: the LG model on the bigger smooth space remembers the singular fibre.

Idea. The kernel is $\mathcal{O}_{f^{-1}(0) \times \mathbb{C}}$ with the Koszul factorization of zf ; it is fully faithful and hits a generator.



3 *Hypersurfaces and pairs of pants*

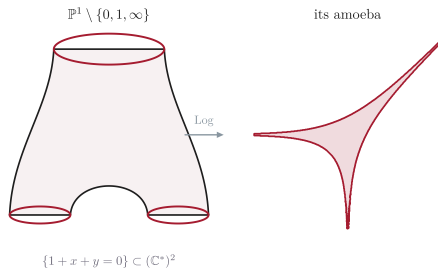
Very affine hypersurfaces and their mirrors

A *very affine hypersurface* is $H = \{f = 0\} \subset (\mathbb{C}^*)^n$, f a Laurent polynomial; its Newton polytope with the subdivision from the coefficients gives a toric variety X .

Gammage–Shende 2022; Zhou

$\mathcal{W}(H) \simeq \text{Coh}(D)$, $D \subset X$ the toric boundary divisor.

Pairs of pants are the atoms. H decomposes into pairs of pants $\Pi^{n-1} = \{1 + z_1 + \cdots + z_n = 0\}$ [Mikhalkin].



$\Pi^1 = \mathbb{P}^1$ minus three points, and its amoeba.

The pair of pants and the node

The pair of pants $\Pi^n = \{z_1 + \cdots + z_{n+1} + 1 = 0\} \subset (\mathbb{C}^*)^{n+1}$ has as intrinsic mirror the *singular* hypersurface $\{z_1 \cdots z_{n+1} = 0\} \subset \mathbb{C}^{n+1}$ [Abouzaid–Auroux–Efimov–Katzarkov–Orlov; Nadler].

$n = 1$. $\Pi^1 = \mathbb{C}^* \setminus \{-1\}$ is mirror to the nodal conic $\{xy = 0\} \subset \mathbb{C}^2$: two lines meeting in a point.

The problem. Mirror symmetry should be an involution, so we also want the *Fukaya category of the node*. But the node is singular: there is no symplectic manifold to build it from.

The way out [Auroux]. A singular fibre $f^{-1}(0)$ comes with smooth nearby fibres $f^{-1}(t)$, and the monodromy around $t = 0$ remembers what degenerates. Use them.

4

The Fukaya category of a singular hypersurface

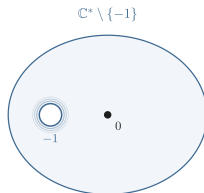
Definition (Auroux)

Let $f : X \rightarrow \mathbb{C}$ have a single singular fibre, over 0. Then

$$D\mathcal{W}(f^{-1}(0)) := D\mathcal{W}(f^{-1}(t)) [s^{-1}],$$

the localization of a nearby fibre's category at Seidel's $s : \mu \rightarrow \text{id}$.

The node, computed. $\mathcal{W}(\{xy = t\}) \simeq \mathbb{C}[z^{\pm 1}]$ -mod (a cylinder). The Dehn twist acts by $s = 1 + z$; inverting it gives $\mathbb{C}[z^{\pm 1}, (1 + z)^{-1}]$, functions on $\mathbb{C}^* \setminus \{-1\} = \Pi^1$: the mirror.



$$\mathbb{C}[z^{\pm 1}] [(1 + z)^{-1}]$$

$$s = 1 + z$$

monodromy transformation

Theorem (Jeffs 2023, Thm. 1.3.1)

Let X be a smooth affine variety with a Stein embedding $X \subset \mathbb{C}^N$ and $f : X \rightarrow \mathbb{C}$ the restriction of a polynomial with a single critical fibre $f^{-1}(0)$. Then for small $t \neq 0$

$$D^\pi \mathcal{W}(f^{-1}(t)) [s^{-1}] \simeq D^\pi \mathcal{W}(X \times \mathbb{C}, zf).$$

This is the symplectic mirror of Orlov's $D^b \text{Coh}(f^{-1}(0)) \simeq \text{MF}(X \times \mathbb{C}, zf)$, conjectured by Orlov.

Step 1: AAK equivalence. Thimbles over Lagrangians in $f^{-1}(t)$: $\mathcal{W}(f^{-1}(t)) \simeq \mathcal{W}(X \times \mathbb{C}, z(f - t))$ [AAK; Jeffs 1.3.4].

Steps 2–3: stop removal. $z(f - t) \rightsquigarrow zf$ removes a stop: a quotient by linking disks [GPS], which split-generate the same subcategory as the cones of s .

Theorem (Jeffs, Thm. 1.4.2)

Suppose HMS holds for $X(B)$ and $X(\check{B})$ so that the monodromy functor μ around the large complex structure limit of $X(\check{B})$ is $\otimes \mathcal{L}^{-1}$ on $X(B)$, with $c_1(\mathcal{L}) = [\omega]$. Then

$$D^\pi \mathcal{F}(X_0(\check{B})) \simeq D^b \text{Coh}(X(B) \setminus s^{-1}(0)), \quad s \in H^0(\mathcal{L}).$$

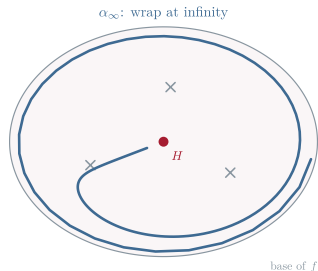
Why localization. Inverting $s : \mu \rightarrow \text{id}$ on the A-side is inverting the section s on the B-side: localising a module category at a section is restricting to the complement of its zero set.

Corollary

The Fukaya category of an elliptic curve with n nodes (a cycle of n rational curves) is equivalent to $D^b \text{Coh}$ of an elliptic curve with n punctures.

HMS equivalences are not unique; ask them to intertwine natural functors. Around a very affine hypersurface $H = f^{-1}(0)$ and its complement $(\mathbb{C}^*)^n \setminus H$ (itself very affine, mirror to the toric boundary Z of the canonical bundle K_X) there are four symplectic functors:

- ρ : restriction to the ends along H ;
- j : lifting along a ray to infinity;
- α_0, α_∞ : wrapping around H / at infinity.



The theorem: the functors are mirror to geometry

Theorem (Gammage–Jeffs; Jeffs 2023, Part II)

Under the HMS equivalences of Kuwagaki and Gammage–Shende, composed with Knörrer periodicity:

- α_0 is mirror to i_{X*} for the inclusion $i_X : X \rightarrow Z$;
- α_∞ is mirror to $\pi_X^*[1]$ for the projection $\pi_X : Z \rightarrow X$;
- ρ is mirror to the quotient $q : \text{Coh}(Z) \rightarrow \text{Sing}(Z)$ (with its $\mathbb{G}_m$ -graded lift);
- j is mirror to $j_{D*}p_D^*$ for $j_D : K_X|_D \rightarrow Z$, $p_D : K_X|_D \rightarrow D$;

and there is an exact triangle $j \cap \rightarrow \alpha_0 \rightarrow \alpha_\infty \xrightarrow{+1}$.

A surprise. The complement has *two* natural mirrors, one of them a derived scheme, related by a non-geometric equivalence through Knörrer periodicity; Auroux's conjectures hold after adjusting for gradings.

Summary of the two lectures

Calabi–Yau

Floer theory and coherent sheaves; HMS for the elliptic curve; SYZ fibrations and tropical limits.

Beyond

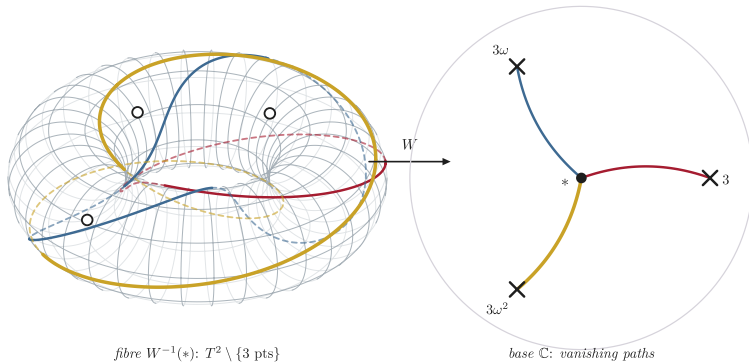
Fano $\leftrightarrow$ LG models; thimbles and exceptional collections; matrix factorizations and Knörrer periodicity; hypersurfaces built from pairs of pants.

Singular fibres [Jeffs 2023]

$\mathcal{W}(f^{-1}(0)) := \mathcal{W}(f^{-1}(t))[s^{-1}]$ satisfies A-side Knörrer periodicity and the expected mirror symmetry at large complex structure limits; and HMS for very affine hypersurfaces respects the natural functors.

Open. Complete intersections (Conjecture 1.3.3), an intrinsic definition from log structures or (-1) -shifted symplectic structures.

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Thank you.
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