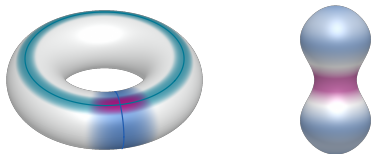


Differential Topology

Volume and de Rham Cohomology: Degree, Gauss–Bonnet and Poincaré Duality

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ETH Zürich



1. Lie derivatives and Cartan's formula
2. Top-degree forms and the degree
3. Volume forms
4. De Rham cohomology and homotopy
5. The Mayer–Vietoris sequence
6. Compact supports and Poincaré duality

Where we stand

From the last lecture

- Differential forms $\Omega^k(M)$, the pullback, the exterior derivative with $d^2 = 0$.
- Partitions of unity on every manifold.
- For oriented M : the integral $\int_M \omega$ of compactly supported top forms.
- Stokes: $\int_M d\omega = \int_{\partial M} \omega$.

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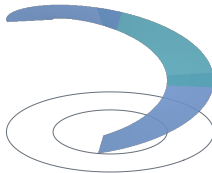
Open questions

- (a) Stokes: exact top forms integrate to 0. Is every top form with $\int = 0$ exact?
- (b) Is a volume form determined, up to diffeomorphism, by its total volume?
- (c) Which closed forms are exact? How large is $H^k(M)$, and how do we compute it?

Why it is hard: local primitives do not glue

- On a disc every closed form has a primitive (we will prove it).
- On an annulus, $d\theta$ has a primitive on every sector; the primitives agree on overlaps up to constants.
- Going round once, the constants add up to 2π : there is no global primitive.

Exactness is a global question; it is decided by how local pieces fit together.



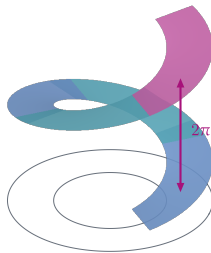
$d\theta_k = d\theta$ on each sector

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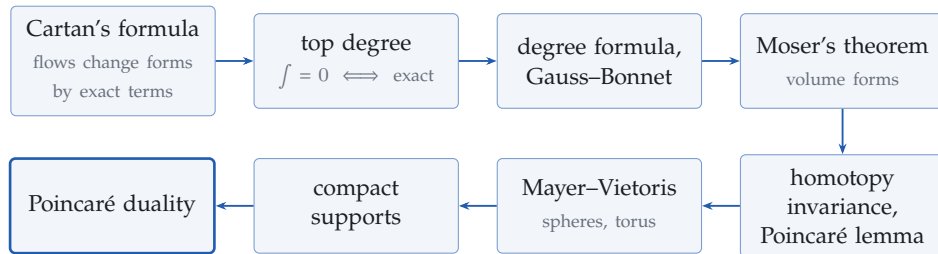
$d\theta_k = d\theta$ on each sector

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$\theta_5 = \theta_1 + 2\pi$

no global θ with $d\theta$

The tools of this lecture



1

*Lie derivatives and Cartan's
formula*

- A vector field X on M (compactly supported, or M compact) has a flow φ_t : $\frac{d}{dt}\varphi_t = X \circ \varphi_t$, $\varphi_0 = \text{id}$, $\varphi_{s+t} = \varphi_s \circ \varphi_t$.
- A smooth family X_t generates an **isotopy** ψ_t : $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, $\psi_0 = \text{id}$; each ψ_t is a diffeomorphism.
- Conversely every isotopy starting at id is generated this way, by $X_t = \left(\frac{d}{dt}\psi_t\right) \circ \psi_t^{-1}$.

The interior product

Definition 1.1 (Interior product)

For $X \in \text{Vect}(M)$ and $\omega \in \Omega^k(M)$ let $(\iota_X \omega)(v_2, \dots, v_k) = \omega(X, v_2, \dots, v_k)$; $\iota_X f = 0$.

Lemma 1.2 (Rules for ι_X)

$\iota_X \iota_X = 0$ and $\iota_X(\omega \wedge \tau) = \iota_X \omega \wedge \tau + (-1)^k \omega \wedge \iota_X \tau$ for $\omega \in \Omega^k$.

Proof The first is skew-symmetry. Both sides of the second are bilinear and pointwise, so take $\omega = dx^I$, $\tau = dx^J$; then it is the expansion of the determinant $dx^I \wedge dx^J(X, \dots)$ along its first column.

The Lie derivative

Definition 1.3 (Lie derivative)

$L_X \omega = \left. \frac{d}{dt} \right|_{t=0} \varphi_t^* \omega$ for the flow φ_t of X . For functions, $L_X f = df(X)$.

Example 1.4 (Area in the plane)

For $\omega = dx \wedge dy$ and $X = (a, b)$: $L_X \omega = (\partial_x a + \partial_y b) dx \wedge dy = (\operatorname{div} X) \omega$.

Rules for the Lie derivative

Lemma 1.5 (Rules for L_X)

$$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*L_X\omega, \quad L_X(\omega \wedge \tau) = L_X\omega \wedge \tau + \omega \wedge L_X\tau, \quad L_Xd = dL_X.$$

Proof $\varphi_{t+s}^* = \varphi_t^*\varphi_s^*$, so $\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\frac{d}{ds}|_0\varphi_s^*\omega$. The product rule follows from $\varphi_t^*(\omega \wedge \tau) = \varphi_t^*\omega \wedge \varphi_t^*\tau$, and $L_Xd = dL_X$ from $\varphi_t^*d = d\varphi_t^*$ (differentiate in t ; d commutes with $\frac{d}{dt}$ in coordinates).

Dragging an area form

$X = \text{swirl} + \text{source}, \operatorname{div} X = 0.56$



tile area $\times 1.40$, $e^{0.56t} = 1.40$

$X = \text{rotation}, \operatorname{div} X = 0$



tile area $\times 1.00$

The tiles of $dx \wedge dy$ carried by two flows: with a source their areas grow like $e^{t \operatorname{div} X}$; under a rotation they only turn. $L_X \omega = (\operatorname{div} X) \omega$.

Dragging an area form

$X = \text{swirl} + \text{source}, \operatorname{div} X = 0.56$



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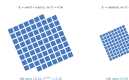
Cartan's formula

Theorem 1.6 (Cartan)

For every vector field X : $L_X = d \iota_X + \iota_X d$ on $\Omega^*(M)$.

1

Same rules. Both L_X and $P = d\iota_X + \iota_X d$ are local, linear, commute with d ($dP = d\iota_X d = Pd$ by $d^2 = 0$), and are derivations: $P(\omega \wedge \tau) = P\omega \wedge \tau + \omega \wedge P\tau$, because the antiderivations d and ι_X of degrees $+1$ and -1 produce cross terms $(-1)^k d\omega \wedge \iota_X \tau$ etc. that cancel in pairs.



tiles dragged by a flow

2

Same on functions. $Pf = \iota_X df = df(X) = L_X f$.

3

Same everywhere. Locally $\omega = \sum f_I dx^{i_1} \wedge \dots \wedge dx^{i_k}$ with $dx^i = d(x^i)$. An operator that is a derivation and commutes with d is determined by its values on functions: $Q(f dx^{i_1} \wedge \dots) = Qf dx^{i_1} \wedge \dots + \sum f \dots \wedge dQx^{i_j} \wedge \dots$.

Two consequences

Corollary 1.7 (d of a 1-form, invariantly)

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

Proof By Cartan, $(L_X \alpha)(Y) = d(\alpha(X))(Y) + d\alpha(X, Y)$. By the product rule for L_X applied to $\alpha(Y) = \iota_Y \alpha$: $X(\alpha(Y)) = (L_X \alpha)(Y) + \alpha(L_X Y)$ with $L_X Y = [X, Y]$. Combine the two.

Corollary 1.8 (Closed forms along flows)

If $d\omega = 0$, then $L_X \omega = d(\iota_X \omega)$ is exact; hence $\varphi_t^* \omega - \omega = d \int_0^t \varphi_s^* \iota_X \omega \, ds$.

Proof Cartan with $d\omega = 0$, then integrate $\frac{d}{ds} \varphi_s^* \omega = \varphi_s^* L_X \omega = d(\varphi_s^* \iota_X \omega)$ from 0 to t .

The homotopy formula

Theorem 1.9 (Homotopy formula)

Let $F : [0, 1] \times M \rightarrow N$ be smooth and $f_t = F(t, \cdot)$. There is a linear $h : \Omega^k(N) \rightarrow \Omega^{k-1}(M)$ with

$$f_1^* \omega - f_0^* \omega = d h \omega + h d \omega.$$

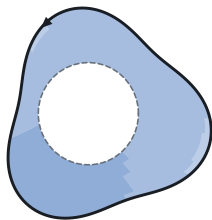
Proof Write $F^* \omega = \alpha_t + dt \wedge \beta_t$ with α_t, β_t containing no dt ; then $\alpha_t = f_t^* \omega$ and

$$d(F^* \omega) = d_M \alpha_t + dt \wedge (\partial_t \alpha_t - d_M \beta_t).$$

Put $h \omega = \int_0^1 \beta_t dt$. Since $F^* d \omega = d(F^* \omega)$, the dt -component of $F^* d \omega$ is $\partial_t \alpha_t - d_M \beta_t$, hence

$$h d \omega = \int_0^1 (\partial_t \alpha_t - d_M \beta_t) dt = \alpha_1 - \alpha_0 - d h \omega = f_1^* \omega - f_0^* \omega - d h \omega.$$

The swept ribbon



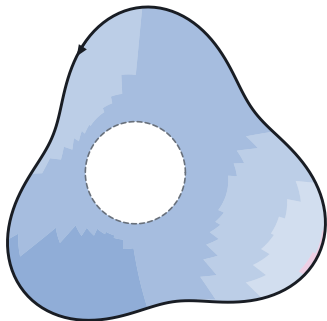
$$\int_{\gamma_t} \alpha - \int_{\gamma_0} \alpha = +4.152$$

$$\int_{\text{ribbon}} d\alpha = +4.153$$

blue: $d\alpha > 0$, purple: $d\alpha < 0$

For a 1-form the homotopy formula is Stokes on the ribbon swept by the loops: $\int_{\gamma_t} \alpha - \int_{\gamma_0} \alpha = \int_{\text{ribbon}} d\alpha$.

The swept ribbon



$$\int_{\gamma_t} \alpha - \int_{\gamma_0} \alpha = +10.156$$

$$\int_{\text{ribbon}} d\alpha = +10.158$$

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For a 1-form the homotopy formula is Stokes on the ribbon swept by the loops: $\int_{\gamma_t} \alpha - \int_{\gamma_0} \alpha = \int_{\text{ribbon}} d\alpha$.

Isotopies with time-dependent fields

Theorem 1.10 (Time-dependent Lie derivative)

Let ψ_t be generated by X_t and ω_t a smooth family of forms. Then

$$\frac{d}{dt} \psi_t^* \omega_t = \psi_t^* \left(L_{X_t} \omega_t + \frac{\partial \omega_t}{\partial t} \right).$$

Proof The $\partial_t \omega_t$ term is the product rule. For a fixed form, $D_t = \frac{d}{dt} \circ \psi_t^*$ and $\psi_t^* \circ L_{X_t}$ are both ψ_t^* -derivations commuting with d , and on functions both give $df(X_t) \circ \psi_t$ by the chain rule. The argument of Cartan's proof, step 3, shows they agree.

Corollary 1.11 (Primitives along isotopies)

If ω is closed and the X_t are compactly supported, then $\psi_1^* \omega - \omega = d \int_0^1 \psi_t^* \iota_{X_t} \omega dt$, a primitive with compact support whenever ω has one. In particular isotopic diffeomorphisms act equally on cohomology.

2 *Top-degree forms and the degree*

Top-degree forms with zero integral

Theorem 2.1 (Exactness in top degree)

Let M be connected, oriented, without boundary, and $\omega \in \Omega_c^m(M)$. Then

$$\int_M \omega = 0 \quad \Longleftrightarrow \quad \omega = d\tau \text{ for some } \tau \in \Omega_c^{m-1}(M).$$

- $(\Leftarrow)$ is Stokes' theorem.
- $(\Rightarrow)$ in two moves: first prove it for forms supported in one cube; then move every piece of ω into the same cube with a compactly supported isotopy, which changes it only by exact forms.

The cube lemma

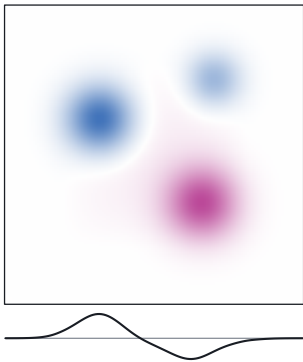
Lemma 2.2 (Cube lemma)

If $f \in C_c^\infty((0, 1)^m)$ and $\int f = 0$, then $f = \sum_{i=1}^m \partial_i u_i$ with $u_i \in C_c^\infty((0, 1)^m)$. Hence

$$f \, dx^1 \wedge \cdots \wedge dx^m = d\left(\sum_i (-1)^{i-1} u_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m\right).$$

Proof Induction on m . For $m = 1$, $u(t) = \int_0^t f$ vanishes near 0, and near 1 because $\int f = 0$. Step: fix $e \in C_c^\infty((0, 1))$ with $\int e = 1$ and put $g(x') = \int_0^1 f(x', t) \, dt$, so $g \in C_c^\infty((0, 1)^{m-1})$ with $\int g = 0$. Then $h = f - g(x')e(x^m)$ integrates to 0 on every vertical line, so $u_m(x', x^m) = \int_0^{x^m} h(x', s) \, ds$ has compact support and $\partial_m u_m = h$. By induction $g = \sum_{i < m} \partial_i v_i$, so $g e = \sum_{i < m} \partial_i (v_i e)$, and $f = \sum_{i < m} \partial_i (v_i e) + \partial_m u_m$.

The cube lemma, seen



columns: $g(x) = \int_0^1 f(x, y) dy$

The column integrals g are subtracted along a fixed bump $e(y)$; then every column has total 0, and its antiderivative from the bottom is zero again at the top.

The cube lemma, seen



$h = f - g(x)e(y)$: every column balances

The column integrals g are subtracted along a fixed bump $e(y)$; then every column has total 0, and its antiderivative from the bottom is zero again at the top.

The cube lemma, seen



$$u(x, y) = \int_0^y h: \text{ zero on } \partial, \partial_y u = h$$

$$f dx \wedge dy = d\left(\sum_i (-1)^{i-1} u_i \widehat{dx^i}\right)$$

u_i compactly supported in the cube

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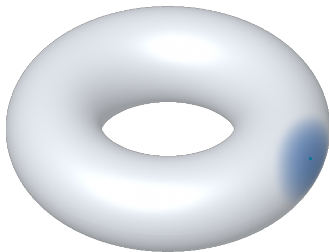
Moving a point anywhere

Lemma 2.3 (Homogeneity)

If M is connected and without boundary, then for $p, q \in M$ there is a compactly supported isotopy ψ_t with $\psi_0 = \text{id}$ and $\psi_1(p) = q$.

Proof Write $p \sim q$ if such an isotopy exists: an equivalence relation (compose and reverse isotopies). Each class is open: in a chart ball around p take $X = \chi \cdot (q' - p)$ with $\chi = 1$ on the segment $[p, q']$ and compactly supported in the ball; its time-1 flow moves p to q' . The complement of a class is a union of classes, also open. M is connected, so there is one class.

Sliding a bump



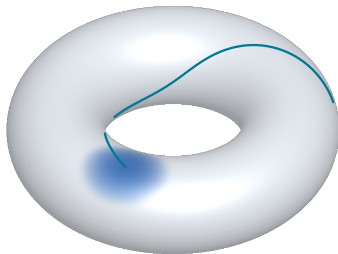
$$\psi_t^* \omega - \omega = d \left(\int_0^t \psi_s^* \iota_{X_s} \omega \, ds \right)$$

$$\int_{T^2} \omega = 0.320$$

the flow is supported near the path

A compactly supported flow carries a bump form along a path; its integral does not change, and $\psi_t^* \omega - \omega$ is exact with a compactly supported primitive.

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Proof of exactness in top degree



a bump carried along

1 **Cut.** Fix an oriented chart $U_0 \cong (0, 1)^m$ and $p_0 \in U_0$. For each p choose ψ_p (homogeneity) with $\psi_p(p_0) = p$ and let $U_p = \psi_p(U_0)$. Finitely many $U_1, \dots, U_n$ cover $\text{supp } \omega$; a partition of unity for $\{M \setminus \text{supp } \omega, U_1, \dots, U_n\}$ gives $\omega = \sum_{i \geq 1} \rho_i \omega$ with $\text{supp } \rho_i \omega \subset U_i$.

2 **Move into the cube.** $\psi_i^*(\rho_i \omega)$ is supported in U_0 , and $\psi_i^*(\rho_i \omega) - \rho_i \omega = d\tau_i$ with τ_i compactly supported (primitives along isotopies). Since ψ_i preserves orientation, $\sigma = \sum_i \psi_i^*(\rho_i \omega)$ has $\int \sigma = \int \omega = 0$.

3 **Cube lemma.** $\sigma = d\tau_0$ with τ_0 compactly supported in U_0 , so $\omega = \sigma - \sum_i (\psi_i^*(\rho_i \omega) - \rho_i \omega) = d(\tau_0 - \sum_i \tau_i)$.

Corollary 2.4 (Integration is an isomorphism)

For M compact, connected, oriented and without boundary, $\int_M : H^m(M) \rightarrow \mathbb{R}$ is an isomorphism.

Proof Well defined by Stokes; onto, since a bump form in one chart has integral 1; injective by the theorem.

Example 2.5 (Non-compact manifolds)

On $M = \mathbb{R}^m$ the form $f dx$ with $f > 0$ compactly supported is exact as a form (every closed form on $\mathbb{R}^m$ is, Section 4), but not as a compactly supported form: compact support matters in the theorem.

The degree formula

Theorem 2.6 (Degree formula)

Let $f : M \rightarrow N$ be smooth, M, N compact, oriented, without boundary, of dimension m , N connected. For all $\omega \in \Omega^m(N)$:

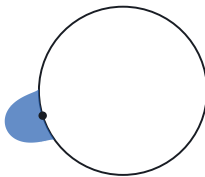
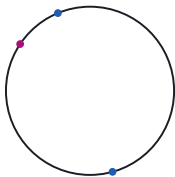
$$\int_M f^* \omega = \deg(f) \int_N \omega.$$

Proof Let q be a regular value with preimages $p_1, \dots, p_n$ and signs $\varepsilon_i = \operatorname{sgn} \det df(p_i)$, $\deg f = \sum \varepsilon_i$. By the inverse function theorem and compactness there is a neighbourhood V of q with $f^{-1}(V) = U_1 \sqcup \dots \sqcup U_n$ and $f : U_i \rightarrow V$ a diffeomorphism of sign ε_i . For ω supported in V : $\int_M f^* \omega = \sum_i \varepsilon_i \int_{U_i} \omega$. For general ω pick ω' supported in V with the same integral; then $\omega - \omega' = d\tau$ and $\int_M f^* d\tau = \int_M d(f^* \tau) = 0$.

Counting with signs

$$f(\theta) = \theta + 1.6 \sin \theta$$

ω near q



$$\int f^* \omega = 0.422$$

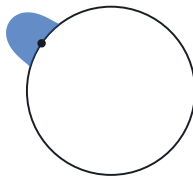
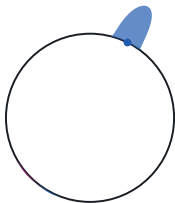
$$\deg f \cdot \int \omega = 1 \times 0.422$$

A bump form near q pulled back: three bumps under $z \mapsto z^3$; under a folded map of degree 1 two blue and one purple, which merge and vanish as q moves past the fold.

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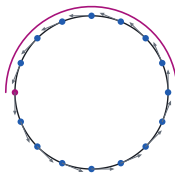
The hairy ball theorem

Corollary 2.7 (Hairy ball)

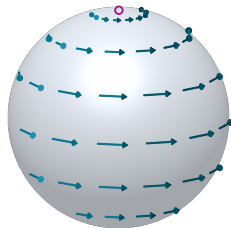
For m even, every vector field on S^m has a zero.

Proof If $v(x) \neq 0$ everywhere, normalise $|v| = 1$. Then $F_t(x) = \cos(\pi t)x + \sin(\pi t)v(x)$ is a homotopy in S^m from id to $a = -\text{id}$, so $\deg a = 1$. But $a^*\sigma = (-1)^{m+1}\sigma$ for $\sigma_x = \det[x \cdot]$, so $\deg a = (-1)^{m+1} = -1$ by the degree formula.

$\text{id} \simeq -\text{id}$



$S^1: x \mapsto \cos(\pi t)x + \sin(\pi t)v(x)$



S^2 : the field vanishes at the poles

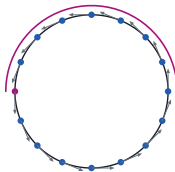
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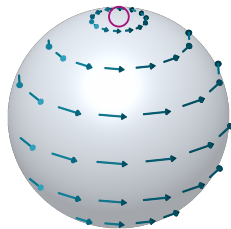
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$S^1: x \mapsto \cos(\pi t)x + \sin(\pi t)v(x)$



S^2 : the field vanishes at the poles

3

Volume forms

The Riemannian volume form

Lemma 3.1 (Volume form of a metric)

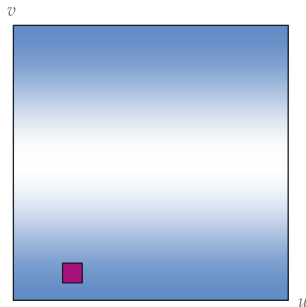
On an oriented Riemannian manifold (M, g) there is a unique m -form dvol_g with value 1 on every positively oriented orthonormal basis. In oriented coordinates $\mathrm{dvol}_g = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^m$.

Proof Two positive orthonormal bases differ by a matrix in $SO(m)$, of determinant 1, so at each point the condition fixes the form, and it exists. In coordinates write $\partial_i = \sum_j A_{ji} e_j$ for a positive orthonormal frame (e_j) ; then $g = A^\top A$, and $\mathrm{dvol}_g(\partial_1, \dots, \partial_m) = \det A = \sqrt{\det g}$: smooth.

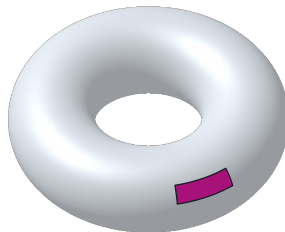
Example 3.2 (Surfaces)

For a parametrised surface with first fundamental form E, F, G : $dA = \sqrt{EG - F^2} \, du \wedge dv$.

The area element



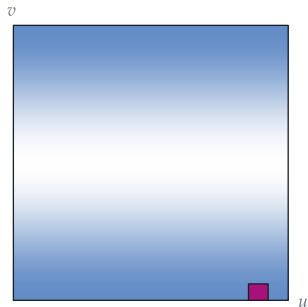
$\sqrt{EG - F^2}$ on the parameters



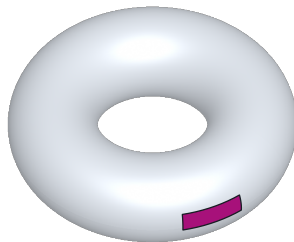
$$\frac{\text{area}}{du dv} = \sqrt{EG - F^2} = 0.61$$

A small parameter square and its image on the torus: on the outer equator the image is large, on the inner one small; $\sqrt{EG - F^2} = r(R + r \cos v)$.

The area element



$\sqrt{EG - F^2}$ on the parameters



$$\frac{\text{area}}{du dv} = \sqrt{EG - F^2} = 0.65$$

A small parameter square and its image on the torus: on the outer equator the image is large, on the inner one small; $\sqrt{EG - F^2} = r(R + r \cos v)$.

Hypersurfaces and the volume of spheres

Example 3.3 (Hypersurfaces)

For an oriented hypersurface $M \subset \mathbb{R}^{m+1}$ with unit normal ν : $(\text{dvol}_M)_p(v_1, \dots, v_m) = \det[\nu(p) v_1 \cdots v_m]$, i.e. $\text{dvol}_M = \iota_\nu(dx^1 \wedge \cdots \wedge dx^{m+1})|_M$.

Proof For a positive orthonormal basis v_i of $T_p M$, the basis $(\nu, v_1, \dots, v_m)$ of $\mathbb{R}^{m+1}$ is orthonormal and positive, so the determinant is 1.

Example 3.4 (Volume of spheres)

$\text{vol}(S^m) = (m+1) \text{vol}(B^{m+1})$: on S^m , $\text{dvol} = \iota_x dV$, and $d(\iota_x dV) = L_x dV = (m+1) dV$ by Cartan, so Stokes on B^{m+1} gives the claim. Hence $\text{vol}(S^1) = 2\pi$, $\text{vol}(S^2) = 4\pi$, $\text{vol}(S^3) = 2\pi^2$.

The Gauss map

Definition 3.5 (Gauss map, Gaussian curvature)

Let $M \subset \mathbb{R}^{m+1}$ be a compact hypersurface bounding a compact domain, with outer unit normal $\nu : M \rightarrow S^m$. Since $T_p M = \nu(p)^\perp = T_{\nu(p)} S^m$, the differential $d\nu(p)$ is an endomorphism of $T_p M$, and $K(p) = \det d\nu(p)$.

Lemma 3.6 (Curvature as an area ratio)

$$\nu^* \mathrm{dvol}_{S^m} = K \, \mathrm{dvol}_M.$$

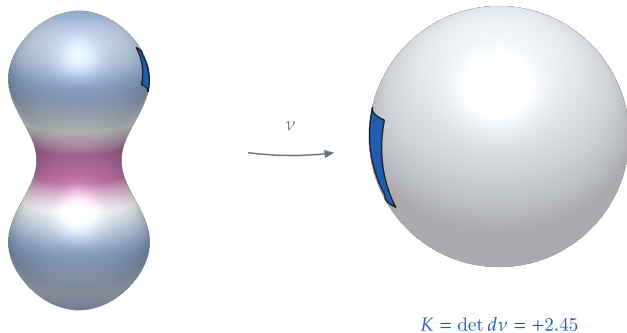
Proof $(\nu^* \mathrm{dvol}_{S^m})_p(v_1, \dots, v_m) = \det[\nu(p) d\nu v_1 \cdots d\nu v_m] = \det d\nu(p) \cdot \det[\nu(p) v_1 \cdots v_m]$, since $d\nu(p)$ maps $\nu^\perp$ to itself and fixes the first column.

Curvature of surfaces

Example 3.7 (Surfaces)

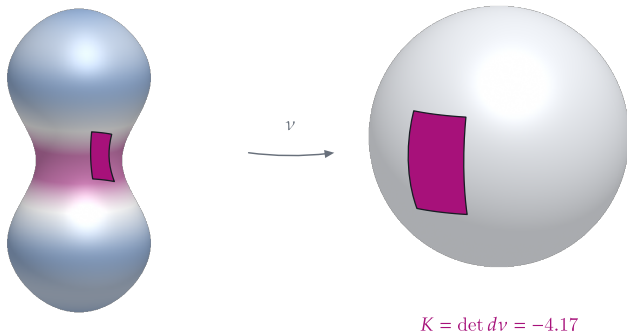
For $m = 2$, $K = (LN - M^2)/(EG - F^2)$; a sphere of radius r has $K = 1/r^2$, a cylinder $K = 0$, a saddle $K < 0$.

The Gauss map, seen



A patch and its image under ν . Where $K > 0$ the image keeps its orientation, where $K < 0$ (the waist) it is reversed; the area ratio tends to $|K|$.

The Gauss map, seen



A patch and its image under ν . Where $K > 0$ the image keeps its orientation, where $K < 0$ (the waist) it is reversed; the area ratio tends to $|K|$.

Theorem 3.8 (Gauss–Bonnet for hypersurfaces)

For m even and $M \subset \mathbb{R}^{m+1}$ compact without boundary:
$$\int_M K \, d\text{vol}_M = \frac{1}{2} \text{vol}(S^m) \chi(M).$$

1

Degree. By the lemma and the degree formula, $\int_M K \, d\text{vol} = \int_M v^* d\text{vol}_{S^m} = \deg(v) \text{vol}(S^m).$

2

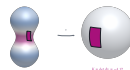
A vector field. Choose $a \in S^m$ with $\pm a$ regular values of v and let $X(p) = a - \langle a, v(p) \rangle v(p)$, the tangential part of a . $X(p) = 0$ iff $v(p) = \pm a$. Since $dX(v) = -\langle a, dv v \rangle v - \langle a, v \rangle dv v$ and $dv v \perp v$, at such p : $dX(p) = \mp dv(p)$.

3

Indices. $\text{ind}_p X = \text{sgn} \det(\mp dv(p)) = \text{sgn} K(p)$ because m is even.

4

Poincaré–Hopf. $\chi(M) = \sum_{v(p)=a} \text{sgn} K + \sum_{v(p)=-a} \text{sgn} K = 2 \deg v$. Insert into step 1.



the Gauss map

Total curvature counts holes

Example 3.9 (Surfaces in $\mathbb{R}^3$)

$\int_M K dA = 2\pi\chi(M) = 4\pi(1 - g)$: 4π for the sphere, 0 for the torus, -4π for genus 2.

On the torus the outer half is curved positively, the inner half negatively, and they cancel exactly.



$$\frac{1}{2\pi} \int K dA = +0.00$$

$$\chi = 0$$

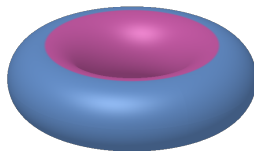
blue: $K > 0$, purple: $K < 0$

Total curvature counts holes

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$$\frac{1}{2\pi} \int K dA = -0.00$$

$$\chi = 0$$

blue: $K > 0$, purple: $K < 0$

Volume forms with the same volume

Theorem 3.10 (Moser)

Let M be compact, connected, oriented, without boundary, and ω_0, ω_1 volume forms with $\int_M \omega_0 = \int_M \omega_1$. Then there is a diffeomorphism ψ , isotopic to id , with $\psi^* \omega_1 = \omega_0$.

Moser's trick Join the forms by $\omega_t = (1 - t)\omega_0 + t\omega_1$ and look for an isotopy with $\psi_t^* \omega_t = \omega_0$. Differentiating turns this non-linear equation for ψ_t into a linear equation for its generating vector field X_t .

Volume forms and vector fields

Lemma 3.11 (Volume forms and vector fields)

For a volume form ω , $X \mapsto \iota_X \omega$ is an isomorphism $\text{Vect}(M) \rightarrow \Omega^{m-1}(M)$.

Proof Pointwise, $v \mapsto \iota_v \omega_p$ is injective (if $v \neq 0$ and $\iota_v \omega_p = 0$, extend v to a basis: $\omega_p = 0$), between spaces of dimension m ; the inverse is smooth by Cramer's rule in a chart.

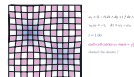
Proof of Moser's theorem

1 **Same orientation.** The set where ω_0, ω_1 have the same sign is open and closed, and nonempty because the integrals agree; M is connected. So every $\omega_t = (1-t)\omega_0 + t\omega_1$ is a volume form.

2 **A primitive.** $\int(\omega_1 - \omega_0) = 0$, so $\omega_1 - \omega_0 = d\tau$ (top cohomology).

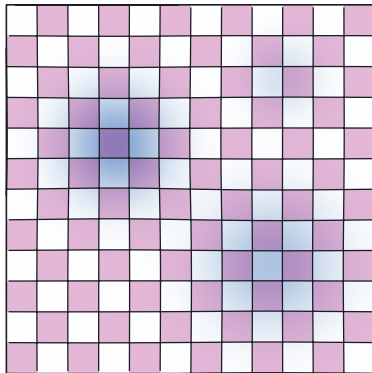
3 **The vector field.** Define X_t by $\iota_{X_t}\omega_t = -\tau$; it depends smoothly on t and its flow ψ_t exists on $[0, 1]$ (M compact).

4 **Constant.** $\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*(d\iota_{X_t}\omega_t + \iota_{X_t}d\omega_t + \omega_1 - \omega_0) = \psi_t^*(d(\iota_{X_t}\omega_t + \tau)) = 0$, so $\psi_1^*\omega_1 = \psi_0^*\omega_0 = \omega_0$.



Moser's flow

Moser's flow on the torus



$$\omega_t = (1 - t) dx \wedge dy + t f dx \wedge dy$$

$$\iota_{X_t} \omega_t = -\tau, \quad d\tau = \omega_1 - \omega_0$$

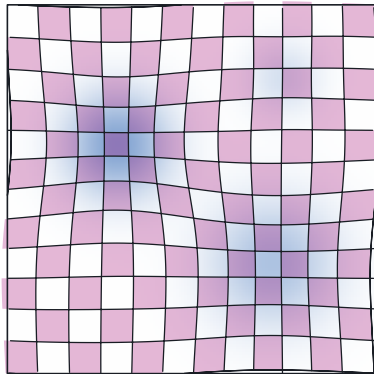
$$t = 0.12$$

$$\text{each cell carries } \omega_t\text{-mass} = \frac{1}{144}$$

shaded: the density f

On $\mathbb{R}^2/\mathbb{Z}^2$: $\tau = \partial_x \varphi dy - \partial_y \varphi dx$ with $\Delta \varphi = f - 1$ (solved by Fourier series), and $X_t = -\nabla \varphi / ((1 - t) + t f)$. Equal squares become cells of equal f -mass.

Moser's flow on the torus



$$\omega_t = (1 - t) dx \wedge dy + t f dx \wedge dy$$

$$\iota_{X_t} \omega_t = -\tau, \quad d\tau = \omega_1 - \omega_0$$

$$t = 1.00$$

$$\text{each cell carries } \omega_t\text{-mass} = \frac{1}{144}$$

shaded: the density f

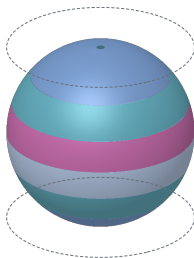
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Archimedes' hat box

Example 3.12 (Archimedes)

The horizontal radial projection of S^2 minus the poles onto the cylinder $S^1 \times (-1, 1)$ preserves area.

Proof Parametrise by $(\theta, z) \mapsto (\sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta, z)$. Then $E = 1 - z^2$, $F = 0$, $G = \frac{1}{1-z^2}$, so $\sqrt{EG - F^2} = 1$ and $dA = d\theta \wedge dz$: the area form of the cylinder in the same coordinates.



$$dA_{S^2} = d\theta \wedge dz$$

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equal heights $\Rightarrow$ equal areas

$$dA_{S^2} = d\theta \wedge dz$$

4

De Rham cohomology and homotopy

The de Rham ring

Definition 4.1 (Recall)

$H^k(M) = \ker d / \operatorname{im} d$ in degree k . A smooth $f : M \rightarrow N$ induces $f^* : H^k(N) \rightarrow H^k(M)$, $[\omega] \mapsto [f^*\omega]$.

Lemma 4.2 (Products)

$[\omega] \cup [\tau] = [\omega \wedge \tau]$ is well defined, making $H^*(M)$ a graded-commutative ring, and f^* is a ring homomorphism.

Proof $d(\omega \wedge \tau) = d\omega \wedge \tau \pm \omega \wedge d\tau = 0$ for closed ω, τ ; and $\omega \wedge d\eta = \pm d(\omega \wedge \eta)$ when $d\omega = 0$, so changing a representative changes the product by an exact form. f^* commutes with d and $\wedge$.

Homotopy invariance

Theorem 4.3 (Homotopy invariance)

If $f_0, f_1 : M \rightarrow N$ are smoothly homotopic, then $f_0^* = f_1^*$ on $H^*(N)$.

Proof For closed ω the homotopy formula gives $f_1^* \omega - f_0^* \omega = d h \omega$.

Homotopy equivalence

Definition 4.4 (Homotopy equivalence)

$f : M \rightarrow N$ is a homotopy equivalence if there is $g : N \rightarrow M$ with $g \circ f \simeq \text{id}_M$ and $f \circ g \simeq \text{id}_N$; M is **contractible** if it is homotopy equivalent to a point.

Corollary 4.5 (Invariance)

Homotopy equivalent manifolds have isomorphic de Rham cohomology.

Proof $f^*g^* = (g \circ f)^* = \text{id}$ and $g^*f^* = \text{id}$ on cohomology.

The Poincaré lemma

Corollary 4.6 (Poincaré lemma)

If M is contractible, then $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k \geq 1$. In particular every closed form on a star-shaped $U \subset \mathbb{R}^m$ is exact.

Proof A point has $H^0 = \mathbb{R}$ and nothing else. For star-shaped U (about 0), $F(t, x) = tx$ is a homotopy from the constant map to id.

The explicit primitive

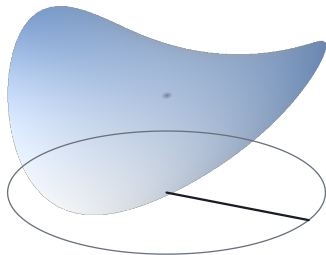
Example 4.7 (The explicit primitive)

The homotopy formula for $F(t, x) = tx$ gives, for a closed k -form ω , the primitive

$$(h\omega)_x(v_2, \dots, v_k) = \int_0^1 t^{k-1} \omega_{tx}(x, v_2, \dots, v_k) dt; \quad k = 1 : f(x) = \int_0^1 \omega_{tx}(x) dt,$$

the line integral along the ray from 0 to x .

Integrating along rays



$$f(x) = \int_0^1 \alpha_{tx}(x) dt$$

$$df = \alpha \text{ where } d\alpha = 0$$

On a disc the integrals along rays assemble into a potential f with $df = \alpha$. On an annulus, for $d\theta$, the same sweep returns 2π higher.

Integrating along rays



$\alpha = d\theta$ on an annulus

back 2π higher: no primitive

On a disc the integrals along rays assemble into a potential f with $df = \alpha$. On an annulus, for $d\theta$, the same sweep returns 2π higher.

Example 4.8 (Punctured space)

$H^k(\mathbb{R}^m \setminus \{0\}) \cong H^k(S^{m-1})$: the inclusion $S^{m-1} \hookrightarrow \mathbb{R}^m \setminus \{0\}$ and $x \mapsto x/|x|$ are homotopy inverse, via $F(t, x) = (1 - t)x + tx/|x|$.

Example 4.9 (Vector bundles, products)

$H^*(M \times \mathbb{R}^n) \cong H^*(M)$, since $M \times \mathbb{R}^n$ retracts onto $M \times \{0\}$ by $F(t, x, y) = (x, ty)$.

Remark 4.10 (Simply connected manifolds). If M is simply connected then $H^1(M) = 0$: path-independent line integrals of a closed 1-form give a primitive. (Stated, not proved here.)

5

The Mayer–Vietoris sequence

Idea Contractible pieces carry no cohomology. To compute H^* of a manifold we glue it from contractible pieces and keep track of the gluing. The bookkeeping is an exact sequence.

Definition 5.1 (Exactness)

A sequence of linear maps $\cdots \rightarrow V_{k-1} \xrightarrow{f_{k-1}} V_k \xrightarrow{f_k} V_{k+1} \rightarrow \cdots$ is **exact** if $\text{im } f_{k-1} = \ker f_k$ for all k . A short exact sequence of complexes is one that is exact in each degree, with maps commuting with d .

Long exact sequences

Lemma 5.2 (Zig-zag)

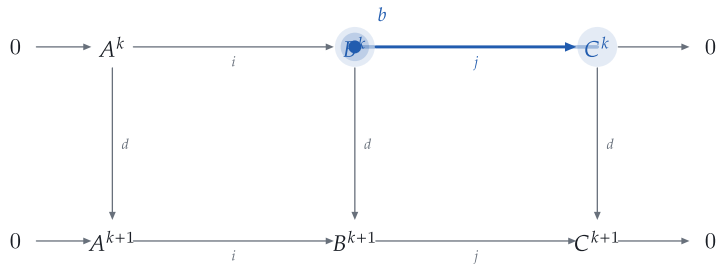
A short exact sequence of cochain complexes $0 \rightarrow A^* \xrightarrow{i} B^* \xrightarrow{j} C^* \rightarrow 0$ gives a long exact sequence

$$\cdots \rightarrow H^k(A) \xrightarrow{i^*} H^k(B) \xrightarrow{j^*} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \xrightarrow{i^*} H^{k+1}(B) \rightarrow \cdots$$

with $\delta[c] = [a]$, where $j(b) = c$ and $i(a) = db$.

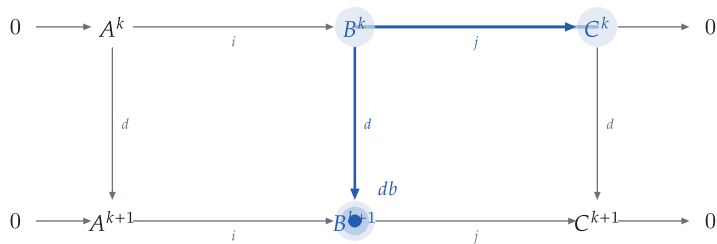
Proof: δ is well defined j is onto, so b exists; $j(db) = dj(b) = dc = 0$, so $db = i(a)$ for a unique a , and $i(da) = d db = 0$ gives $da = 0$. Another lift $b' = b + i(x)$ changes a to $a + dx$; replacing c by $c + dc'$ with $c' = j(b'')$ changes b by db'' and a not at all. So $[a]$ depends only on $[c]$.

The zig-zag, seen



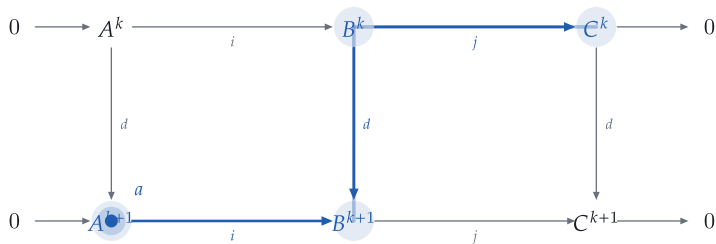
j is onto: lift c to b

The zig-zag, seen



$$j(db) = d j(b) = dc = 0$$

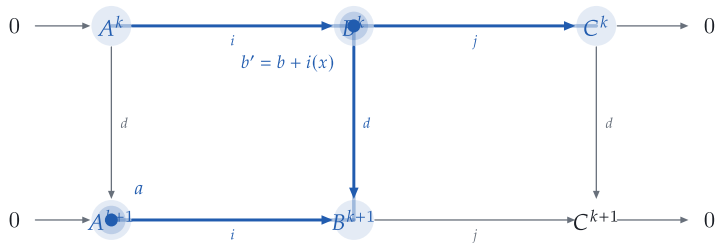
The zig-zag, seen



so $db = i(a)$, and $da = 0$: $\delta[c] = [a]$

The zig-zag, seen

another lift: $a' = a + dx$



so $db = i(a)$, and $da = 0$: $\delta[c] = [a]$

Exactness of the long sequence

Proof: exactness, three places

- **At $H^k(B)$.** $j^*i^* = (ji)^* = 0$. If $j^*[b] = 0$, i.e. $j(b) = dc'$ with $c' = j(b')$, then $j(b - db') = 0$, so $b - db' = i(a)$ with $i(da) = db = 0$, $da = 0$: $[b] = i^*[a]$.
- **At $H^k(C)$.** $\delta j^*[b] = [a]$ with $i(a) = db = 0$, so 0. If $\delta[c] = 0$, i.e. $a = dx$, then $b - i(x)$ is closed and $j(b - i(x)) = c$.
- **At $H^{k+1}(A)$.** $i^*\delta[c] = [db] = 0$. If $i^*[a] = 0$, i.e. $i(a) = db$, then $c = j(b)$ is closed ($dc = j(db) = ji(a) = 0$) and $\delta[c] = [a]$.

The Mayer–Vietoris sequence

Theorem 5.3 (Mayer–Vietoris)

For $M = U \cup V$ with U, V open, the sequence

$$0 \rightarrow \Omega^*(M) \xrightarrow{\omega \mapsto (\omega|_U, \omega|_V)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{(\alpha, \beta) \mapsto \beta|_{U \cap V} - \alpha|_{U \cap V}} \Omega^*(U \cap V) \rightarrow 0$$

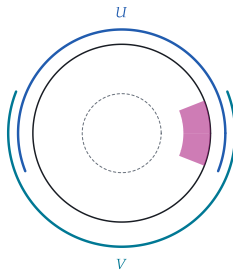
is exact. Hence there is a long exact sequence $\cdots \rightarrow H^k(M) \rightarrow H^k(U) \oplus H^k(V) \rightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \rightarrow \cdots$, with $\delta[\omega] = [d\rho_U \wedge \omega]$ (extended by 0) for a partition of unity (ρ_U, ρ_V) .

Proof Injectivity and exactness in the middle: forms that agree on overlaps glue. Surjectivity: for ω on $U \cap V$, $\rho_V \omega$ extends by 0 to U ($\text{supp } \rho_V \subset V$) and $\rho_U \omega$ to V ; $(-\rho_V \omega, \rho_U \omega) \mapsto (\rho_U + \rho_V)\omega = \omega$. Then $d(-\rho_V \omega, \rho_U \omega) = (-d\rho_V \wedge \omega, d\rho_U \wedge \omega)$ agree on $U \cap V$ since $d\rho_U = -d\rho_V$: this is $\delta[\omega]$.

The circle

Example 5.4 (S^1)

U, V two arcs, $U \cap V$ two arcs. In $0 \rightarrow H^0(S^1) \rightarrow \mathbb{R}^2 \xrightarrow{\delta} \mathbb{R}^2 \rightarrow H^1(S^1) \rightarrow 0$ the middle map $(a, b) \mapsto (b - a, b - a)$ has rank 1. So $H^0 = \mathbb{R}$, and $H^1(S^1) \cong \mathbb{R}^2 / \mathbb{R} = \mathbb{R}$, generated by δ of the function that is 1 on one arc of $U \cap V$.



$f = 1$ on one arc of $U \cap V$, 0 on the other

$df = 0$, but $f \neq g|_{U \cap V} - h|_{U \cap V}$

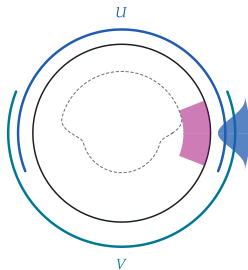
$\delta f = d\rho_U \cdot f$: a bump 1-form

$\int_{S^1} \delta f = 1$: it generates $H^1(S^1)$

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$df = 0$, but $f \neq g|_{U \cap V} - h|_{U \cap V}$

$\delta f = d\rho_U \cdot f$: a bump 1-form

$\int_{S^1} \delta f = 1$: it generates $H^1(S^1)$

Example 5.5 ($S^m, m \geq 1$)

$H^k(S^m) = \mathbb{R}$ for $k = 0, m$ and 0 otherwise.

Proof Induction on m ; $m = 1$ is the circle. Cover S^m by $U = S^m \setminus \{N\}$, $V = S^m \setminus \{S\}$, both diffeomorphic to $\mathbb{R}^m$, with $U \cap V \simeq S^{m-1}$. For $k \geq 1$ the sequence

$$\underbrace{H^k(U) \oplus H^k(V)}_0 \rightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(S^m) \rightarrow \underbrace{H^{k+1}(U) \oplus H^{k+1}(V)}_0$$

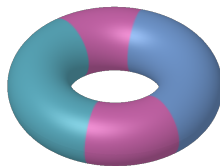
gives $H^{k+1}(S^m) \cong H^k(S^{m-1})$. In degree 1: $0 \rightarrow H^0(S^m) \rightarrow \mathbb{R}^2 \rightarrow H^0(U \cap V) = \mathbb{R} \rightarrow H^1(S^m) \rightarrow 0$ with the middle map onto, so $H^1(S^m) = 0$ for $m \geq 2$; $H^0 = \mathbb{R}$ by connectedness.

The torus

Example 5.6 (T^2)

$$H^0 = \mathbb{R}, H^1 = \mathbb{R}^2, H^2 = \mathbb{R}.$$

Proof Two cylinders $U, V \simeq S^1$, $U \cap V \simeq S^1 \sqcup S^1$. In degrees 0, 1: $0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \xrightarrow{r_0} \mathbb{R}^2 \rightarrow H^1(T^2) \rightarrow \mathbb{R}^2 \xrightarrow{r_1} \mathbb{R}^2 \rightarrow H^2(T^2) \rightarrow 0$, where $r_0, r_1 : (a, b) \mapsto (b-a, b-a)$ have rank 1. Hence $\dim H^1 = 1 + 1 = 2$ and $\dim H^2 = 2 - 1 = 1$.



$$T^2 = U \cup V$$

$$U \simeq V \simeq S^1$$

$$U \cap V \simeq S^1 \sqcup S^1$$

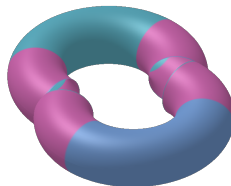
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The torus

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$$H^0 = \mathbb{R}, H^1 = \mathbb{R}^2, H^2 = \mathbb{R}.$$

Proof Two cylinders $U, V \simeq S^1$, $U \cap V \simeq S^1 \sqcup S^1$. In degrees 0, 1: $0 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^2 \xrightarrow{r_0} \mathbb{R}^2 \rightarrow H^1(T^2) \rightarrow \mathbb{R}^2 \xrightarrow{r_1} \mathbb{R}^2 \rightarrow H^2(T^2) \rightarrow 0$, where $r_0, r_1 : (a, b) \mapsto (b-a, b-a)$ have rank 1. Hence $\dim H^1 = 1 + 1 = 2$ and $\dim H^2 = 2 - 1 = 1$.



$$T^2 = U \cup V$$

$$U \simeq V \simeq S^1$$

$$U \cap V \simeq S^1 \sqcup S^1$$

$$H^1(T^2) \cong \mathbb{R}^2$$

Good covers and finiteness

Definition 5.7 (Good cover)

An open cover in which every nonempty finite intersection is diffeomorphic to $\mathbb{R}^m$.

Fact (Riemannian geometry)

Every manifold has a good cover, and a finite one if it is compact. Reason: small geodesic balls of a Riemannian metric are geodesically convex, intersections of convex sets are convex, and a geodesically convex open set is diffeomorphic to $\mathbb{R}^m$.

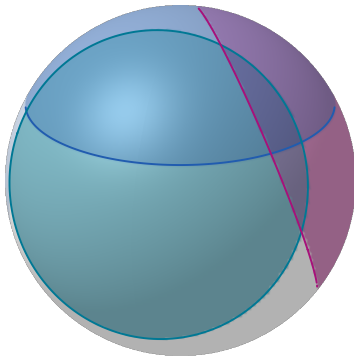
Theorem 5.8 (Finite dimensionality)

If M has a finite good cover, then every $H^k(M)$ is finite-dimensional. In particular this holds for compact M .

Finite dimensionality: proof

Proof Induction on the number p of sets ($p = 1: \mathbb{R}^m$). Let $U = U_1 \cup \dots \cup U_{p-1}$, $V = U_p$; U and $U \cap V$ have good covers with $p - 1$ sets. In $H^{k-1}(U \cap V) \xrightarrow{\delta} H^k(M) \rightarrow H^k(U) \oplus H^k(V)$ the outer spaces are finite-dimensional, hence so is the middle one.

A good cover of the sphere



balls: 4

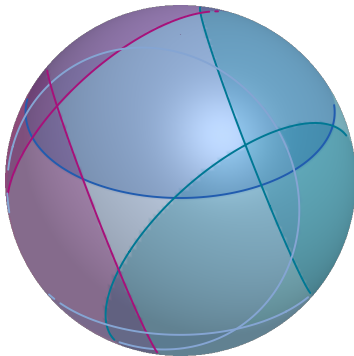
nonempty pairwise intersections: 6

each one convex, so contractible

induction on the number of balls

Geodesic discs of radius below $\pi/2$ are convex; all their intersections are again convex, hence contractible.

A good cover of the sphere



balls: 8

nonempty pairwise intersections: 24

each one convex, so contractible

induction on the number of balls

Geodesic discs of radius below $\pi/2$ are convex; all their intersections are again convex, hence contractible.

6

Compact supports and Poincaré duality

Cohomology with compact supports

Definition 6.1 (Compactly supported cohomology)

$H_c^k(M)$ is the cohomology of the complex $\Omega_c^*(M)$ of compactly supported forms.

- For compact M , $H_c^*(M) = H^*(M)$.
- Open inclusions $U \subset M$ act forwards, $H_c^*(U) \rightarrow H_c^*(M)$ (extend by 0); proper maps act backwards.
- It is *not* homotopy invariant: $H_c^0(\mathbb{R}^m) = 0$, but $H_c^0(\text{pt}) = \mathbb{R}$.

Example 6.2 (Top degree)

For M connected, oriented, without boundary: $\int : H_c^m(M) \rightarrow \mathbb{R}$ is an isomorphism (Section 2).

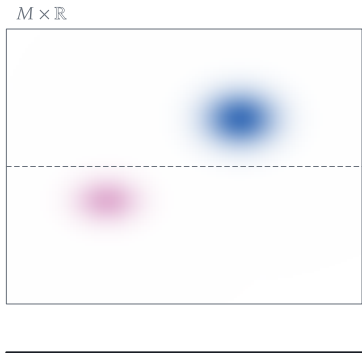
The compact Poincaré lemma

Theorem 6.3 (Integration along the fibre)

For $\pi : M \times \mathbb{R} \rightarrow M$, integration along the fibre π_* induces $H_c^k(M \times \mathbb{R}) \cong H_c^{k-1}(M)$, with inverse $e_* : \varphi \mapsto \pi^* \varphi \wedge e(t) dt$, e a bump with $\int e = 1$. Hence $H_c^k(\mathbb{R}^m) = \mathbb{R}$ for $k = m$ and 0 otherwise.

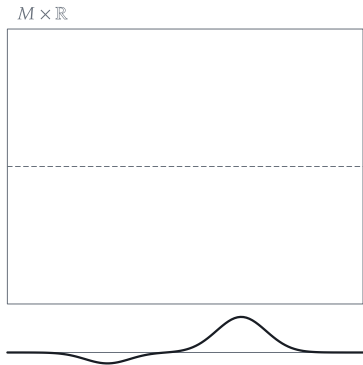
- Every compactly supported form on $M \times \mathbb{R}$ is a sum of forms of type (I) $\pi^* \varphi \cdot f(x, t)$ and type (II) $\pi^* \varphi \cdot f(x, t) dt$.
- π_* kills type (I) and sends type (II) to $\varphi \int_{\mathbb{R}} f(x, t) dt$. It commutes with d , and $\pi_* e_* = \text{id}$.
- It remains to find K with $\text{id} - e_* \pi_* = \pm(dK - Kd)$ on forms of each degree.

Integration along the fibre, seen



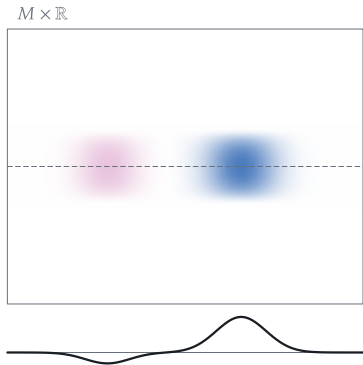
$$\pi_*(f dx \wedge dt) = \left(\int f dt \right) dx$$

Integration along the fibre, seen



$$\pi_*(f dx \wedge dt) = \left(\int f dt \right) dx$$

Integration along the fibre, seen



$$\pi_*(f dx \wedge dt) = \left(\int f dt \right) dx$$

$$e_*(g dx) = g e(t) dx \wedge dt$$

$$\omega - e_*\pi_*\omega = \pm(dK\omega - K d\omega)$$

$$H_c^k(M \times \mathbb{R}) \cong H_c^{k-1}(M)$$

The homotopy K

1

Definition. With $E(t) = \int_{-\infty}^t e$: $K = 0$ on type (I) and $K(\pi^* \varphi f dt) = \pi^* \varphi \cdot g$, $g(x, t) = \int_{-\infty}^t f(x, s) ds - E(t) \int_{-\infty}^{\infty} f(x, s) ds$, compactly supported.



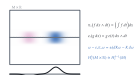
2

Type (II), degree q . $\partial_t g = f - e(t) \int_{\mathbb{R}} f$, and $\pi^* \varphi \int_{\mathbb{R}} f \cdot e dt = e_* \pi_* \omega$, so $dK\omega = d\pi^* \varphi \cdot g + (-1)^{q-1} \pi^* \varphi \wedge (d_x g + \partial_t g dt)$ and $Kd\omega = d\pi^* \varphi \cdot g + (-1)^{q-1} \pi^* \varphi \wedge d_x g$. Hence $dK\omega - Kd\omega = (-1)^{q-1}(\omega - e_* \pi_* \omega)$.



3

Type (I), degree q . $\pi_* \omega = 0$, $K\omega = 0$, and the dt -part of $d\omega$ is $(-1)^q \pi^* \varphi \partial_t f dt$, whose K is $(-1)^q \pi^* \varphi f$ (as $\int_{\mathbb{R}} \partial_t f = 0$). So again $dK\omega - Kd\omega = (-1)^{q-1}(\omega - e_* \pi_* \omega)$.



squeeze and spread

Mayer–Vietoris with compact supports

Theorem 6.4 (Compact Mayer–Vietoris)

For $M = U \cup V$ open, the sequence $0 \rightarrow \Omega_c^*(U \cap V) \xrightarrow{\sigma \mapsto (-\sigma, \sigma)} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{(\alpha, \beta) \mapsto \alpha + \beta} \Omega_c^*(M) \rightarrow 0$ (all maps extension by zero) is exact, giving a long exact sequence $\cdots \rightarrow H_c^k(U \cap V) \rightarrow H_c^k(U) \oplus H_c^k(V) \rightarrow H_c^k(M) \xrightarrow{\delta_c} H_c^{k+1}(U \cap V) \rightarrow \cdots$.

Proof Surjectivity: $\omega = \rho_U \omega + \rho_V \omega$, and $\text{supp}(\rho_U \omega) \subset \text{supp} \rho_U \cap \text{supp} \omega$ is a compact subset of U . Exactness in the middle: if $\alpha + \beta = 0$ then $\alpha = -\beta$ is supported in $\text{supp} \alpha \cap \text{supp} \beta$, a compact subset of $U \cap V$. Injectivity is clear.

Idea The arrows point the other way than for H^* . That is what makes the two sequences dual to each other.

The Poincaré pairing

Definition 6.5 (Poincaré pairing)

For oriented M^m without boundary: $H^k(M) \times H_c^{m-k}(M) \rightarrow \mathbb{R}$, $([\omega], [\tau]) \mapsto \int_M \omega \wedge \tau$.

Well defined $\omega \wedge \tau$ has compact support. Changing representatives: $\int (\omega + d\eta) \wedge \tau - \int \omega \wedge \tau = \int d(\eta \wedge \tau) = 0$ by Stokes, and similarly for τ .

Theorem 6.6 (Poincaré duality)

If M is oriented and has a finite good cover, then $PD : H^k(M) \rightarrow (H_c^{m-k}(M))^*$, $[\omega] \mapsto \int_M \omega \wedge \cdot$, is an isomorphism for every k .

Proof of Poincaré duality

1

$\mathbb{R}^m$. Both sides vanish except $H^0 = \mathbb{R}$ and $(H_c^m)^* = \mathbb{R}$ (compact Poincaré lemma), and $PD(1) = \int$, an isomorphism.



2

The ladder. Write the Mayer–Vietoris sequence of H^* above the dual of the compact one, joined by PD for $M, U, V, U \cap V$. Restriction squares commute because extension by 0 is adjoint to restriction. The δ -square commutes up to sign: with $\delta\omega = d\rho_U \wedge \omega$ and $\delta_c\tau = -d\rho_U \wedge \tau$, $\int_M \delta\omega \wedge \tau = \pm \int_{U \cap V} \omega \wedge \delta_c\tau$.



3

Five lemma. In a commutative ladder of exact rows, if the four outer rungs are isomorphisms so is the middle one (signs do not matter).



4

Induction on the size p of a good cover: with U a union of $p - 1$ sets and V the last one, U, V and $U \cap V$ have good covers of at most $p - 1$ sets; by induction their rungs are isomorphisms, so the rung of M is one.



the ladder

The five lemma

Lemma 6.7 (Five lemma)

In a commutative diagram with exact rows $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5$ and $B_1 \rightarrow \cdots \rightarrow B_5$ and vertical maps f_i : if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3 .

Proof *Injective.* If $f_3(a) = 0$, then $f_4(\partial a) = 0$, so $\partial a = 0$ and $a = \partial a_2$; $f_2(a_2)$ maps to 0, so $f_2(a_2) = \partial b_1 = \partial f_1(a_1)$, hence $a_2 - \partial a_1 \in \ker f_2 = 0$ and $a = \partial \partial a_1 = 0$. *Surjective.* For $b \in B_3$: $\partial b = f_4(a_4)$ with $f_5(\partial a_4) = \partial \partial b = 0$, so $a_4 = \partial a$; then $b - f_3(a) \mapsto 0$, so $b - f_3(a) = \partial b_2 = \partial f_2(a_2) = f_3(\partial a_2)$. (Here ∂ denotes the horizontal maps.)

The duality ladder

each square commutes up to sign

$$\begin{array}{ccccccccc}
 H^{k-1}(U) \oplus H^{k-1}(V) & \longrightarrow & H^{k-1}(U \cap V) & \longrightarrow & H^k(M) & \longrightarrow & H^k(U) \oplus H^k(V) & \longrightarrow & H^k(U \cap V) \\
 \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD \\
 \dots^* & \longrightarrow & H_c^{m-k+1}(U \cap V)^* & \longrightarrow & H_c^{m-k}(M)^* & \longrightarrow & \dots^* & \longrightarrow & H_c^{m-k}(U \cap V)^*
 \end{array}$$

Exact rows joined by the Poincaré maps. Once the four outer rungs are isomorphisms, the five lemma gives the middle one.

The duality ladder

each square commutes up to sign

$$\begin{array}{ccccccccc}
 H^{k-1}(U) \oplus H^{k-1}(V) & \longrightarrow & H^{k-1}(U \cap V) & \longrightarrow & H^k(M) & \longrightarrow & H^k(U) \oplus H^k(V) & \longrightarrow & H^k(U \cap V) \\
 \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD \\
 \dots^* & \longrightarrow & H_c^{m-k+1}(U \cap V)^* & \longrightarrow & H_c^{m-k}(M)^* & \longrightarrow & \dots^* & \longrightarrow & H_c^{m-k}(U \cap V)^*
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Exact rows joined by the Poincaré maps. Once the four outer rungs are isomorphisms, the five lemma gives the middle one.

The duality ladder

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$$\begin{array}{ccccccccc}
 H^{k-1}(U) \oplus H^{k-1}(V) & \longrightarrow & H^{k-1}(U \cap V) & \longrightarrow & H^k(M) & \longrightarrow & H^k(U) \oplus H^k(V) & \longrightarrow & H^k(U \cap V) \\
 \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD & & \downarrow PD \\
 \dots^* & \longrightarrow & H_c^{m-k+1}(U \cap V)^* & \longrightarrow & H_c^{m-k}(M)^* & \longrightarrow & \dots^* & \longrightarrow & H_c^{m-k}(U \cap V)^*
 \end{array}$$

Exact rows joined by the Poincaré maps. Once the four outer rungs are isomorphisms, the five lemma gives the middle one.

Consequences of duality

Corollary 6.8 (Closed manifolds)

For M compact, oriented, without boundary: $H^k(M) \cong H^{m-k}(M)^*$, so the Betti numbers satisfy $b_k = b_{m-k}$. If m is odd, then $\sum_k (-1)^k b_k = 0$.

Proof $H_c = H$ for compact M . For odd m the terms $(-1)^k b_k$ and $(-1)^{m-k} b_{m-k}$ cancel in pairs.

Remark 6.9 (Euler characteristic). $\sum_k (-1)^k b_k$ equals the Euler characteristic of Poincaré–Hopf, and so, by Gauss–Bonnet, $\frac{1}{2\pi} \int_{\Sigma} K dA$ for surfaces. (Stated without proof.)

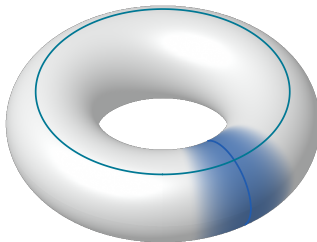
Example 6.10 (Surfaces of genus g)

$b_0 = b_2 = 1$ and $b_1 = 2g$: $\chi = 2 - 2g$, matching $\int K dA = 4\pi(1 - g)$.

Duality as intersection

Example 6.11 (The torus)

For closed curves a, b on T^2 , take bump 1-forms η_a, η_b supported in bands across them, with integral 1 across. Then $\int_{T^2} \eta_a \wedge \eta_b = \pm 1$, the signed number of crossings of a and b : the pairing counts intersections.



η_a : a bump across a

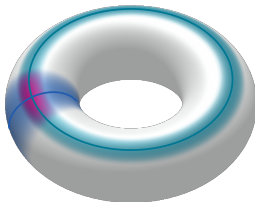
η_b : a bump across b

$$\int_{T^2} \eta_a \wedge \eta_b = 0.00$$

Duality as intersection

Example 6.11 (The torus)

For closed curves a, b on T^2 , take bump 1-forms η_a, η_b supported in bands across them, with integral 1 across. Then $\int_{T^2} \eta_a \wedge \eta_b = \pm 1$, the signed number of crossings of a and b : the pairing counts intersections.



η_a : a bump across a

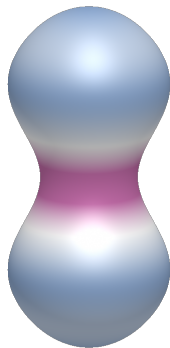
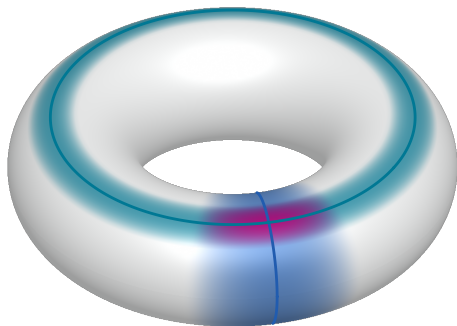
η_b : a bump across b

$$\int_{T^2} \eta_a \wedge \eta_b = 1.00 \\ = a \cdot b, \text{ one crossing}$$

What we proved

1. Cartan: $L_X = d\iota_X + \iota_X d$; the homotopy formula.
2. Top degree: $\int \omega = 0 \iff \omega = d\tau$ (compact supports, connected).
3. Degree formula $\int f^* \omega = \deg f \int \omega$; the hairy ball theorem.
4. Riemannian volume, the Gauss map, Gauss–Bonnet.
5. Moser: volume forms of equal volume are diffeomorphic.
6. Homotopy invariance, the Poincaré lemma.
7. Mayer–Vietoris: spheres, the torus, finite dimensionality.
8. Compact supports and Poincaré duality.

Outlook The same Mayer–Vietoris argument proves the Künneth formula and the Thom isomorphism for vector bundles; the Euler class of the tangent bundle integrates to $\chi(M)$, and Chern–Weil theory writes characteristic classes as curvature forms, of which Gauss–Bonnet is the first instance.



Thank you.