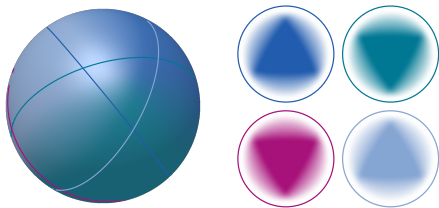


Differential Topology

Integration on Manifolds: Forms, Partitions of Unity and Stokes' Theorem

Johannes Herter

ETH Zürich

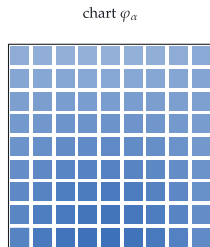


1. Alternating forms
2. Differential forms
3. The exterior derivative
4. Smooth bumps and partitions of unity
5. Orientation
6. The integral
7. Stokes' theorem

What we want to integrate

- The area of a surface, the flux of a field through a sphere, the work along a curve.
- Each is an integral over a **curved** space, computed in coordinates.
- Coordinates are not unique. A plain function gives a different number in every chart.

The integrand has to know how areas change under a change of chart.



$$\sum f |\text{cell}| = 0.698$$

$$F = \varphi_\beta \circ \varphi_\alpha^{-1}$$

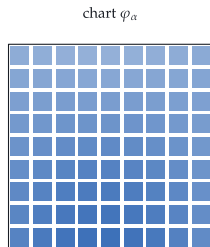


chart φ_β

$$\sum f |F(\text{cell})| = 0.000$$

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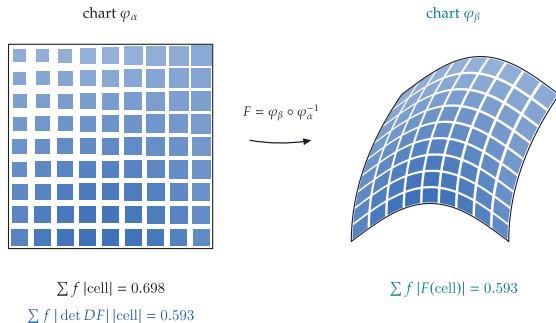


$$\sum f |F(\text{cell})| = 0.593$$

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The integrand has to know how areas change under a change of chart.

What is known, and what is missing

Known

- Integrals $\int_U f \, dx$ over open sets $U \subset \mathbb{R}^m$.
- Change of variables:
$$\int_V f \, dy = \int_U (f \circ F) |\det DF| \, dx.$$
- Smooth manifolds, charts, transition maps, tangent spaces.

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Missing

- (a) An object whose local expression changes by $\det$ of the transition.
- (b) A way to cut a global integrand into chart-sized pieces and add the results.
- (c) A consistent sign: $\det$ versus $|\det|$.

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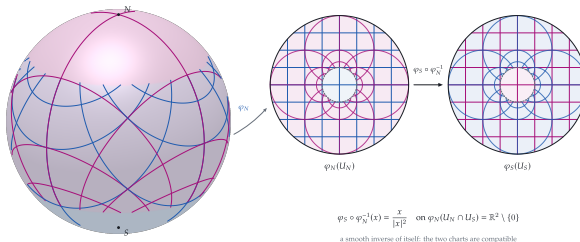
The plan (a) differential forms (b) partitions of unity (c) orientation $\implies \int_M \omega$ is well defined, and Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$.

Why it is hard: there is no global chart

- S^2 needs at least two charts, for instance the stereographic projections from N and S .
- On the overlap the coordinates are related by

$$\varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{|x|^2}.$$

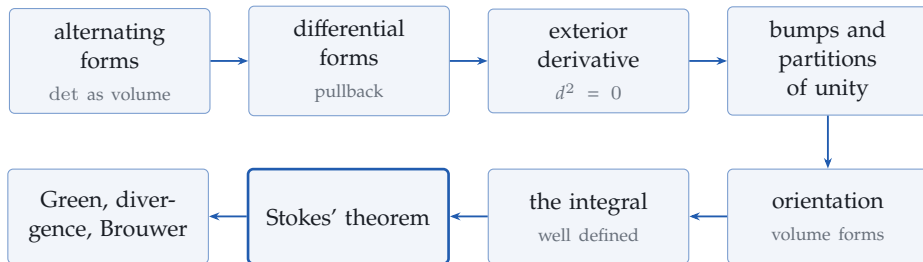
- Its Jacobian determinant is $-1/|x|^4 < 0$: the two charts disagree about orientation.



$$\varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{|x|^2} \text{ on } \varphi_N(U_N \cap U_S) = \mathbb{R}^2 \setminus \{0\}$$

a smooth inverse of itself: the two charts are compatible

The road through this lecture



1 *Alternating forms*

Alternating multilinear forms

Definition 1.1 (Alternating k -form)

Let V be a real vector space of dimension m and $k \geq 1$. An **alternating k -form** on V is a multilinear map $\omega : V \times \cdots \times V \rightarrow \mathbb{R}$ (k factors) with

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) \omega(v_1, \dots, v_k) \quad \text{for all } \sigma \in S_k.$$

They form a vector space $\Lambda^k V^*$. By convention $\Lambda^0 V^* = \mathbb{R}$.

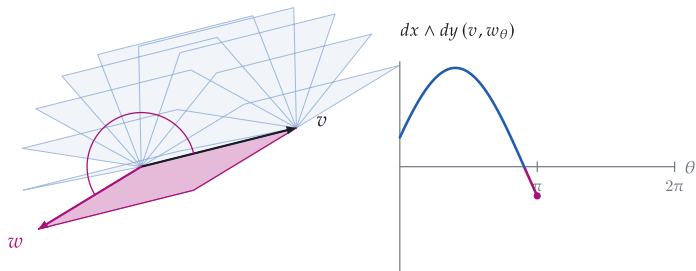
Example 1.2 (Small degrees)

$\Lambda^1 V^* = V^*$, the linear functionals. A 2-form is a skew bilinear form, e.g. $(v, w) \mapsto v^1 w^2 - v^2 w^1$ on $\mathbb{R}^2$.

Example 1.3 (Top degree)

On $\mathbb{R}^m$ the determinant $(v_1, \dots, v_m) \mapsto \det[v_1 \cdots v_m]$ is an alternating m -form.

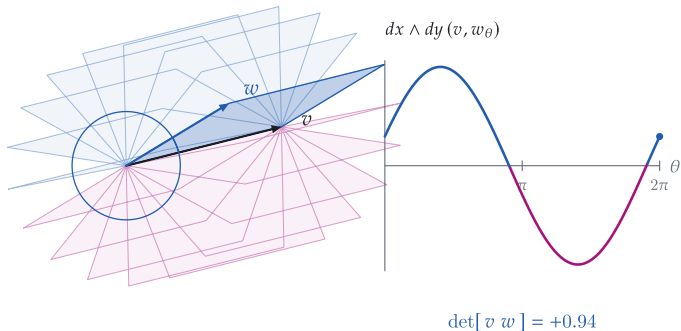
A 2-form measures oriented area



$$\det[v \ w] = -0.94$$

$dx \wedge dy(v, w) = \det[v \ w] = |v| |w| \sin \theta$: positive while w lies to the left of v , negative to its right.

A 2-form measures oriented area



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Three ways to say alternating

Lemma 1.4 (Equivalent conditions)

For a multilinear $\omega : V^k \rightarrow \mathbb{R}$ the following are equivalent:

- (i) ω is alternating;
- (ii) $\omega(v_1, \dots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$;
- (iii) $\omega(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k$ are linearly dependent.

Idea Condition (iii) is the useful one: an alternating form only sees the parallelepiped spanned by its arguments, and a degenerate one has no volume.

Three ways to say alternating: proof

Proof

- (i) $\Rightarrow$ (ii): swapping the two equal entries changes nothing, yet multiplies by -1 , so $\omega = -\omega = 0$ there.
- (ii) $\Rightarrow$ (i): $0 = \omega(\dots, v + w, \dots, v + w, \dots) = \omega(\dots, v, \dots, w, \dots) + \omega(\dots, w, \dots, v, \dots)$, so a transposition flips the sign. Transpositions generate S_k , and sgn is multiplicative.
- (ii) $\Rightarrow$ (iii): write one vector as $v_j = \sum_{i \neq j} c_i v_i$ and expand; every term has a repeated entry.
- (iii) $\Rightarrow$ (ii): a tuple with a repeated entry is linearly dependent.

A basis: the forms dx^I

Definition 1.5 (Elementary forms)

For an increasing multi-index $I = (i_1 < \dots < i_k)$ in $\{1, \dots, m\}$ let

$$dx^I(v_1, \dots, v_k) = \det (v_j^{i_l})_{l,j=1}^k,$$

the $k \times k$ minor of the matrix $[v_1 \cdots v_k]$ in the rows I . Here $dx^i(v) = v^i$.

Lemma 1.6 (Basis)

The forms dx^I , I increasing, are a basis of $\Lambda^k(\mathbb{R}^m)^*$, so $\dim \Lambda^k(\mathbb{R}^m)^* = \binom{m}{k}$.

A basis: proof

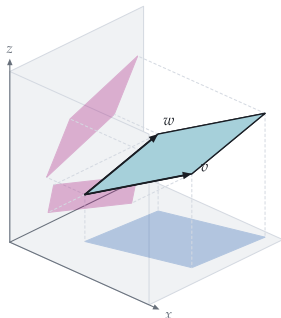
Proof Spanning. By multilinearity, ω is determined by its values $\omega(e_{j_1}, \dots, e_{j_k})$. By (ii) and the sign rule only increasing J matter, and $\omega = \sum_I \omega(e_{i_1}, \dots, e_{i_k}) dx^I$: both sides agree on every $(e_{j_1}, \dots, e_{j_k})$. *Independence.* $dx^I(e_{j_1}, \dots, e_{j_k}) = \delta_{IJ}$ for increasing I, J .

Top degree and the shadows of a parallelogram

Corollary 1.7 (Top and beyond)

$\Lambda^m V^*$ is one-dimensional, and $\Lambda^k V^* = 0$ for $k > m$. In a basis every alternating m -form is a multiple of $\det$.

Proof $\binom{m}{m} = 1$, and for $k > m$ any k vectors are dependent, so (iii) applies.



signed areas of the shadows

$$dx \wedge dy(v, w) = +1.19$$

$$dy \wedge dz(v, w) = -0.43$$

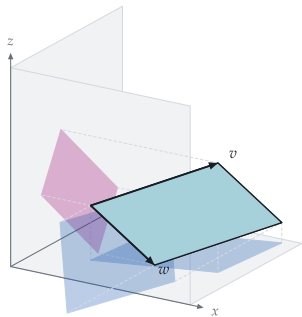
$$dz \wedge dx(v, w) = -0.38$$

Top degree and the shadows of a parallelogram

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signed areas of the shadows

$$dx \wedge dy(v, w) = +1.19$$

$$dy \wedge dz(v, w) = -0.87$$

$$dz \wedge dx(v, w) = +1.12$$

The wedge product

Definition 1.8 (Exterior product)

For $\omega \in \Lambda^k V^*$ and $\tau \in \Lambda^l V^*$ define $\omega \wedge \tau \in \Lambda^{k+l} V^*$ by

$$(\omega \wedge \tau)(v_1, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \tau(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

Example 1.9 (Two covectors)

$(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v)$. In particular $dx \wedge dy(v, w) = v^1 w^2 - v^2 w^1 = \det[v \ w]$.

Idea The factor $1/(k!l!)$ removes the repetitions: permuting inside the first k slots or inside the last l slots does not change a term.

The algebra of the wedge product

Lemma 1.10 (Rules)

The wedge product is bilinear and associative, it satisfies $\omega \wedge \tau = (-1)^{kl} \tau \wedge \omega$ for $\omega \in \Lambda^k V^*$, $\tau \in \Lambda^l V^*$, and $dx^{i_1} \wedge \cdots \wedge dx^{i_k} = dx^I$ for $I = (i_1 < \cdots < i_k)$.

Proof, part 1 It is alternating. Precompose with a transposition ρ and substitute $\sigma \mapsto \sigma\rho$: the sum is unchanged except for $\text{sgn}(\sigma\rho) = -\text{sgn}(\sigma)$. *Bilinearity* is clear from the formula.

Graded commutativity. Let $v \in S_{k+l}$ move the last l slots in front of the first k , keeping their order. It is a product of kl transpositions, so $\text{sgn } v = (-1)^{kl}$. Substituting $\sigma \mapsto \sigma v$ in the sum for $\tau \wedge \omega$ turns it into the sum for $\omega \wedge \tau$, up to the factor $\text{sgn } v$.

The algebra of the wedge product, continued

Proof, part 2 Associativity. For ω, τ, η of degrees k, l, n both $(\omega \wedge \tau) \wedge \eta$ and $\omega \wedge (\tau \wedge \eta)$ equal

$$\frac{1}{k!l!n!} \sum_{\sigma \in S_{k+l+n}} \text{sgn}(\sigma) \omega(v_{\sigma(1..k)}) \tau(v_{\sigma(k+1..k+l)}) \eta(v_{\sigma(k+l+1..k+l+n)}).$$

Expanding $(\omega \wedge \tau)$ inside the outer sum produces each term of this sum exactly $(k+l)!$ times, and $\frac{1}{(k+l)!n!} \cdot \frac{(k+l)!}{k!l!} = \frac{1}{k!l!n!}$. The other bracketing is the same computation.

Elementary forms. By induction on k : $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ evaluated on $(v_1, \dots, v_k)$ is the sum over $\sigma \in S_k$ of $\text{sgn } \sigma \prod_l v_{\sigma(l)}^{i_l}$, which is the Leibniz formula for the minor dx^I .

Wedge products in $\mathbb{R}^3$

Example 1.11 (Cross product and triple product)

For $\alpha = a_1 dx + a_2 dy + a_3 dz$ and $\beta = b_1 dx + b_2 dy + b_3 dz$:

$$\alpha \wedge \beta = (a_2 b_3 - a_3 b_2) dy \wedge dz + (a_3 b_1 - a_1 b_3) dz \wedge dx + (a_1 b_2 - a_2 b_1) dx \wedge dy,$$

the components of $a \times b$. For a third covector γ : $\alpha \wedge \beta \wedge \gamma = \det[a \ b \ c] dx \wedge dy \wedge dz$, the triple product.

Example 1.12 (Repeated factors)

$\alpha \wedge \alpha = -\alpha \wedge \alpha = 0$ for every 1-form; but for 2-forms, $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ on $\mathbb{R}^4$ has $\omega \wedge \omega = 2 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0$.

Idea Graded commutativity only forces $\omega \wedge \omega = 0$ in odd degree.

Pulling back along a linear map

Definition 1.13 (Pullback)

For a linear map $\Phi : V \rightarrow W$ and $\omega \in \Lambda^k W^*$ let $(\Phi^* \omega)(v_1, \dots, v_k) = \omega(\Phi v_1, \dots, \Phi v_k)$.

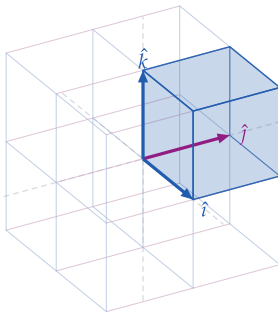
Lemma 1.14 (Pullback rules)

Φ^* is linear, $\Phi^*(\omega \wedge \tau) = \Phi^* \omega \wedge \Phi^* \tau$ and $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$. If $\dim V = m$ and $\Phi : V \rightarrow V$, then for every $\omega \in \Lambda^m V^*$

$$\Phi^* \omega = \det(\Phi) \omega.$$

Proof The first three statements follow by inserting the definitions. For the last one: $\Phi^* \omega$ lies in the one-dimensional space $\Lambda^m V^*$, so $\Phi^* \omega = c \omega$. Choose a basis and identify $V = \mathbb{R}^m$; then $\omega = a \det$, and $\Phi^* \det(e_1, \dots, e_m) = \det[\Phi e_1 \cdots \Phi e_m] = \det \Phi$. So $c = \det \Phi$.

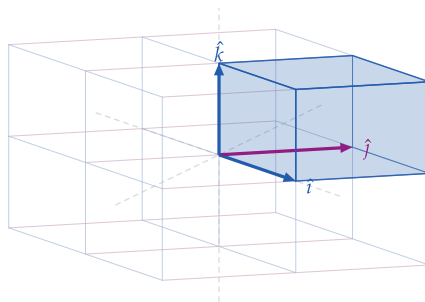
Volume is a determinant



$$\det = 1.00$$

$dx \wedge dy \wedge dz (Ae_1, Ae_2, Ae_3) = \det A$: the cube's image has volume $|\det A|$; the sign records whether A keeps or reverses orientation.

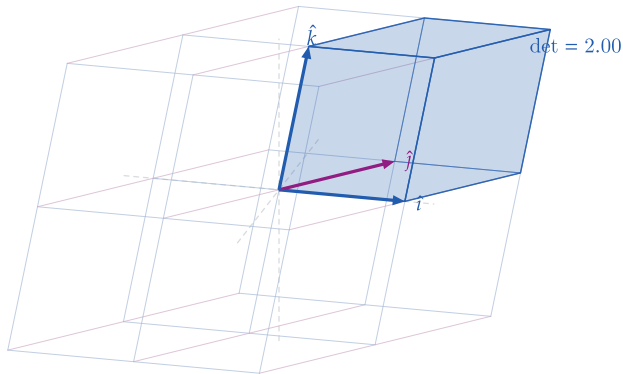
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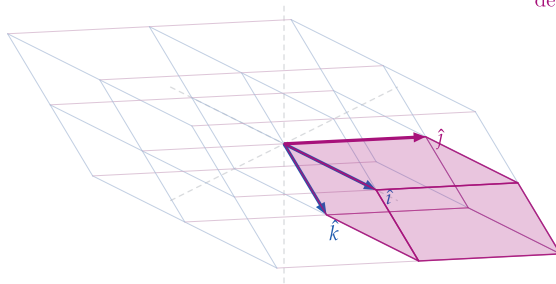
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Volume is a determinant



$$\det = -1.29$$

$dx \wedge dy \wedge dz (Ae_1, Ae_2, Ae_3) = \det A$: the cube's image has volume $|\det A|$; the sign records whether A keeps or reverses orientation.

Two consequences

Corollary 1.15 (Volumes scale by the determinant)

For $\omega \in \Lambda^m(\mathbb{R}^m)^*$ and $A \in \mathbb{R}^{m \times m}$: $\omega(Av_1, \dots, Av_m) = \det(A) \omega(v_1, \dots, v_m)$. Moreover $\det(AB) = \det A \det B$.

Proof The first identity is $A^* \omega = \det(A) \omega$. For the second, $(AB)^* = B^* A^*$, so $\det(AB) \omega = B^*(A^* \omega) = \det A \det B \omega$ with $\omega = \det \neq 0$.

Idea This is the property we were missing in (a): an alternating m -form changes under a linear change of variables exactly by the determinant, with its sign.

2

Differential forms

Differential forms on a manifold

Definition 2.1 (Differential k -form)

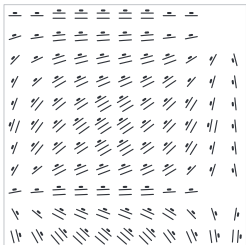
A **k -form** on a smooth m -manifold M assigns to each $p \in M$ an alternating form $\omega_p \in \Lambda^k T_p^* M$, smoothly: in every chart $\varphi : U \rightarrow \mathbb{R}^m$ with coordinates $x^1, \dots, x^m$,

$$\omega = \sum_I \omega_I dx^I \quad \text{with smooth coefficients } \omega_I.$$

The k -forms make up the space $\Omega^k(M)$; $\Omega^0(M) = C^\infty(M)$.

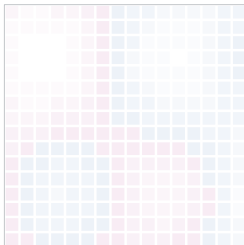
- The wedge product is taken pointwise: $\Omega^k \times \Omega^l \rightarrow \Omega^{k+l}$; $f\omega = f \wedge \omega$ for functions f .
- Smoothness does not depend on the chart: the coefficients in two charts are related by the smooth Jacobian of the transition (slide after next).

What a form looks like



$$\alpha = a \, dx + b \, dy$$

a 1-form: level strokes



$$\omega = f \, dx \wedge dy$$

a 2-form: signed tiles

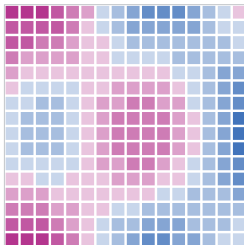
A 1-form is a field of covectors, drawn by its level strokes (dense where it is large); a 2-form is a density with a sign.

What a form looks like



$$\alpha = a dx + b dy$$

a 1-form: level strokes

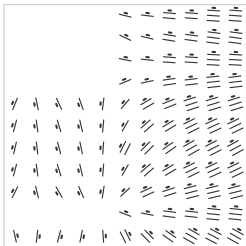


$$\omega = f dx \wedge dy$$

a 2-form: signed tiles

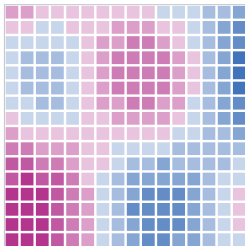
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What a form looks like



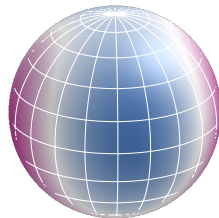
$$\alpha = a \, dx + b \, dy$$

a 1-form: level strokes



$$\omega = f \, dx \wedge dy$$

a 2-form: signed tiles



$$\omega \in \Omega^2(S^2)$$

a top form on a surface

A 1-form is a field of covectors, drawn by its level strokes (dense where it is large); a 2-form is a density with a sign.

Pulling back along a smooth map

Definition 2.2 (Pullback)

For a smooth $F : M \rightarrow N$ and $\omega \in \Omega^k(N)$ let

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p v_1, \dots, dF_p v_k), \quad F^*f = f \circ F.$$

Lemma 2.3 (Rules for the pullback)

$F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ and $(G \circ F)^* = F^*G^*$. In coordinates y on N ,

$$F^*(f dy^{j_1} \wedge \dots \wedge dy^{j_k}) = (f \circ F) dF^{j_1} \wedge \dots \wedge dF^{j_k}, \quad dF^j = \sum_i \frac{\partial F^j}{\partial x^i} dx^i,$$

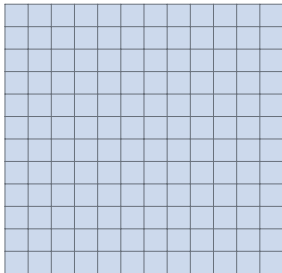
and for open $U, V \subset \mathbb{R}^m$ and $F : U \rightarrow V$: $F^*(f dy^1 \wedge \dots \wedge dy^m) = (f \circ F) \det DF dx^1 \wedge \dots \wedge dx^m$.

Pulling back along a smooth map: proof

Proof Pointwise, F^* is the linear pullback by dF_p , so the linear rules apply; the chain rule $d(G \circ F)_p = dG_{F(p)} dF_p$ gives functoriality. $dy^j \circ dF_p = dF_p^j$ gives the coordinate formula, and the last formula is $\Phi^* \omega = \det \Phi \omega$ for $\Phi = DF(x)$.

Pulling back an area form

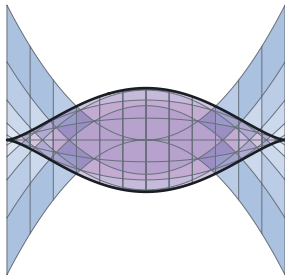
$$F(x, y) = (x, y^3 - (1 - x^2)y)$$



Left: the cells of a grid under a fold map, shaded by $\det DF$; where $\det DF < 0$ the image is folded over. Right: polar coordinates wrap a rectangle onto the disc.

Pulling back an area form

$$F(x, y) = (x, y^3 - (1 - x^2)y)$$

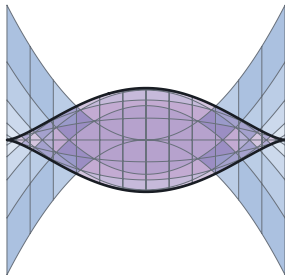


$$\det DF = 3y^2 - 1 + x^2; \text{ blue } > 0, \text{ purple } < 0$$

Left: the cells of a grid under a fold map, shaded by $\det DF$; where $\det DF < 0$ the image is folded over. Right: polar coordinates wrap a rectangle onto the disc.

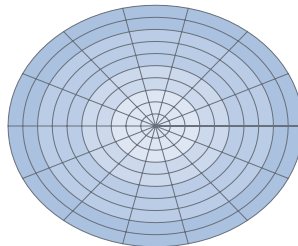
Pulling back an area form

$$F(x, y) = (x, y^3 - (1 - x^2)y)$$



$$\det DF = 3y^2 - 1 + x^2; \text{ blue } > 0, \text{ purple } < 0$$

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$



$$dx \wedge dy = r dr \wedge d\theta$$

Left: the cells of a grid under a fold map, shaded by $\det DF$; where $\det DF < 0$ the image is folded over. Right: polar coordinates wrap a rectangle onto the disc.

Two coordinate computations

Example 2.4 (Polar coordinates)

For $F(r, \theta) = (r \cos \theta, r \sin \theta)$: $dF^1 = \cos \theta dr - r \sin \theta d\theta$, $dF^2 = \sin \theta dr + r \cos \theta d\theta$, so

$$F^*(dx \wedge dy) = (\cos \theta \cdot r \cos \theta + r \sin \theta \cdot \sin \theta) dr \wedge d\theta = r dr \wedge d\theta.$$

Example 2.5 (The sphere)

Let $\sigma = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ on $\mathbb{R}^3$ and $F(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. A direct expansion gives

$$F^*\sigma = \sin \theta d\theta \wedge d\phi.$$

In units of $d\theta \wedge d\phi$: $dy \wedge dz \mapsto \sin^2 \theta \cos \phi$, $dz \wedge dx \mapsto \sin^2 \theta \sin \phi$, $dx \wedge dy \mapsto \sin \theta \cos \theta$; then $\sin^3 \theta \cos^2 \phi + \sin^3 \theta \sin^2 \phi + \cos^2 \theta \sin \theta = \sin \theta$.

One form, many charts

Lemma 2.6 (Transition rule)

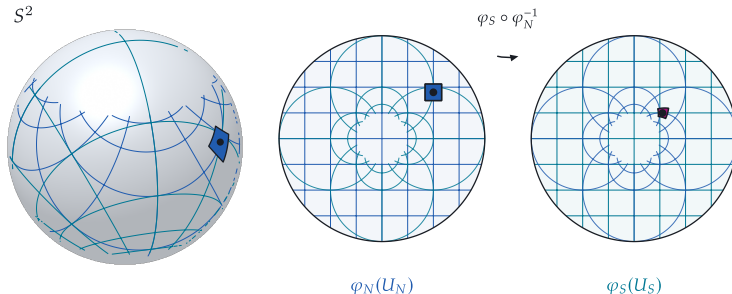
Let $\varphi_\alpha, \varphi_\beta$ be charts and $\omega_\alpha = (\varphi_\alpha^{-1})^* \omega$, $\omega_\beta = (\varphi_\beta^{-1})^* \omega$ the local expressions of $\omega \in \Omega^k(M)$. On $\varphi_\alpha(U_\alpha \cap U_\beta)$:

$$\omega_\alpha = (\varphi_\beta \circ \varphi_\alpha^{-1})^* \omega_\beta.$$

Conversely, local forms ω_α on $\varphi_\alpha(U_\alpha)$ that satisfy this rule come from exactly one $\omega \in \Omega^k(M)$.

Proof $\varphi_\alpha^{-1} = \varphi_\beta^{-1} \circ (\varphi_\beta \circ \varphi_\alpha^{-1})$ on the overlap, and pullback reverses composition. Conversely, set $\omega_p := (\varphi_\alpha)_p^*(\omega_\alpha)$ for any α with $p \in U_\alpha$; the rule says the choice of α does not matter, and smoothness is local.

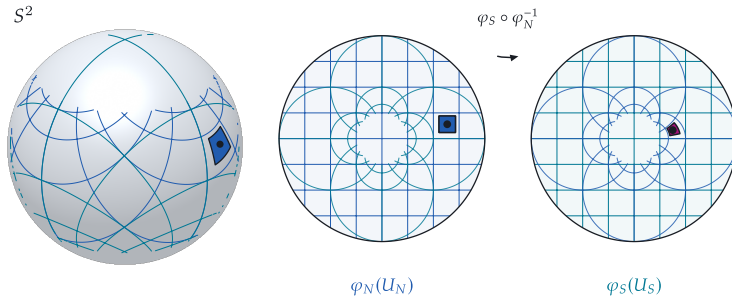
Two charts on the sphere



$$\det D(\varphi_S \circ \varphi_N^{-1}) = -\frac{1}{|x|^4} = -0.31$$

For top forms the transition rule reads $f_\alpha = (f_\beta \circ \tau) \det D\tau$: the coefficient changes by $\det$. Here a small square in the north chart is always turned inside out in the south chart.

Two charts on the sphere



$$\det D(\varphi_S \circ \varphi_N^{-1}) = -\frac{1}{|x|^4} = -0.35$$

For top forms the transition rule reads $f_\alpha = (f_\beta \circ \tau) \det D\tau$: the coefficient changes by $\det$. Here a small square in the north chart is always turned inside out in the south chart.

The stereographic transition

Example 2.7 (S^2 with two charts)

With $\varphi_N(p) = \frac{(p_1, p_2)}{1-p_3}$ and $\varphi_S(p) = \frac{(p_1, p_2)}{1+p_3}$:

$$\tau(x) = \varphi_S \circ \varphi_N^{-1}(x) = \frac{x}{|x|^2} \quad (x \neq 0), \quad \det D\tau(x) = -\frac{1}{|x|^4}.$$

Hence $\tau^*(dy^1 \wedge dy^2) = -|x|^{-4} dx^1 \wedge dx^2$. Composing φ_S with the reflection $(y^1, y^2) \mapsto (y^1, -y^2)$ makes the determinant $+|x|^{-4}$.

Proof $D\tau(x) = |x|^{-2}(I - 2\hat{x}\hat{x}^T)$ with $\hat{x} = x/|x|$; the bracket is the reflection in $\hat{x}^\perp$, with determinant -1 , and the factor contributes $|x|^{-4}$ in dimension 2.

We return to this in Section 5: an atlas with only positive Jacobians is an orientation.

3

The exterior derivative

Differentiating forms

- For a function, $df = \sum_i \partial_i f dx^i$ is the linear part of f at each point.
- For a k -form we want a $(k + 1)$ -form that measures how ω fails to cancel around the boundary of a small $(k + 1)$ -cell.
- It must be **natural**: compatible with pullback, so that it makes sense on manifolds.

Definition 3.1 (d on open sets of $\mathbb{R}^m$)

For $\omega = \sum_I \omega_I dx^I \in \Omega^k(U)$ define

$$d\omega = \sum_I d\omega_I \wedge dx^I = \sum_I \sum_{i=1}^m \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^I \in \Omega^{k+1}(U).$$

Gradient, curl and divergence

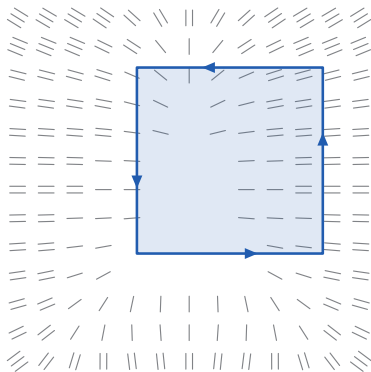
Example 3.2 ($\mathbb{R}^3$)

Identify $f \leftrightarrow f$, $F \leftrightarrow F_1 dx + F_2 dy + F_3 dz$, $F \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$, $f \leftrightarrow f dx \wedge dy \wedge dz$. Then

$$\begin{array}{lll} d : \Omega^0 \rightarrow \Omega^1 & f \mapsto \partial_x f dx + \partial_y f dy + \partial_z f dz & \nabla f \\ d : \Omega^1 \rightarrow \Omega^2 & F \cdot dx \mapsto (\text{curl } F) \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy) & \text{curl } F \\ d : \Omega^2 \rightarrow \Omega^3 & \mapsto (\partial_x F_1 + \partial_y F_2 + \partial_z F_3) dx \wedge dy \wedge dz & \text{div } F \end{array}$$

Idea $d \circ d = 0$ becomes $\text{curl } \nabla f = 0$ and $\text{div } \text{curl } F = 0$. Both come from the symmetry of second derivatives.

d measures circulation per area



$$d\alpha_p(e_1, e_2) = 1.354$$

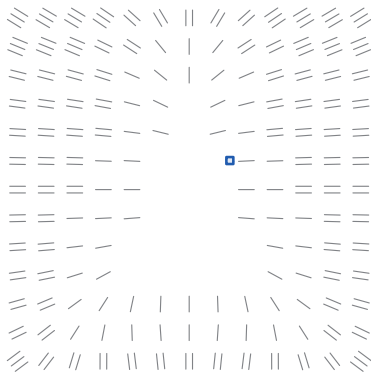


$$\frac{1}{h^2} \oint_{\partial Q_h} \alpha = 1.469$$

$$h = 1.60$$

For $\alpha = a dx + b dy$: $\frac{1}{h^2} \oint_{\partial Q_h} \alpha \rightarrow (\partial_x b - \partial_y a)(p) = d\alpha_p(e_1, e_2)$ as the square Q_h shrinks to p .

d measures circulation per area



$$d\alpha_p(e_1, e_2) = 1.354$$



$$\frac{1}{h^2} \oint_{\partial Q_h} \alpha = 1.354$$

$$h = 0.06$$

For $\alpha = a dx + b dy$: $\frac{1}{h^2} \oint_{\partial Q_h} \alpha \rightarrow (\partial_x b - \partial_y a)(p) = d\alpha_p(e_1, e_2)$ as the square Q_h shrinks to p .

The rules of d

Lemma 3.3 (Properties of d on $\mathbb{R}^m$)

For open $U \subset \mathbb{R}^m$:

- (i) d is linear;
- (ii) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau$ for $\omega \in \Omega^k(U)$;
- (iii) $d \circ d = 0$;
- (iv) $F^*d\omega = dF^*\omega$ for every smooth $F : U \rightarrow V$ between open sets.

Proof of (i), (ii) (i) is clear. For (ii) it suffices, by bilinearity, to take $\omega = f dx^I$, $\tau = g dx^J$. Then $\omega \wedge \tau = fg dx^I \wedge dx^J$ and

$$d(\omega \wedge \tau) = (g df + f dg) \wedge dx^I \wedge dx^J = d\omega \wedge \tau + (-1)^k f dx^I \wedge dg \wedge dx^J,$$

moving the 1-form dg past the k -form dx^I costs $(-1)^k$.

The rules of d , continued

Proof of (iii) For a function,

$$d(df) = \sum_{i,j} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0,$$

since the coefficients are symmetric in (i, j) and $dx^j \wedge dx^i$ is skew. Next $d(dx^I) = 0$ by definition (constant coefficient). For $\omega = f dx^I$, (ii) gives $d(d\omega) = d(df \wedge dx^I) = d(df) \wedge dx^I - df \wedge d(dx^I) = 0$.

Proof of (iv) For functions this is the chain rule: $F^*df = d(f \circ F)$. For $\omega = f dy^{j_1} \wedge \cdots \wedge dy^{j_k}$,

$$F^*\omega = (f \circ F) dF^{j_1} \wedge \cdots \wedge dF^{j_k},$$

and by (ii) and (iii), $d(F^*\omega) = d(f \circ F) \wedge dF^{j_1} \wedge \cdots \wedge dF^{j_k}$, since every $d(dF^j) = 0$. This equals $F^*(df) \wedge F^*dy^{j_1} \wedge \cdots = F^*(d\omega)$.

A computation

Example 3.4 (The angle form is closed)

With $r^2 = x^2 + y^2$,

$$d\left(\frac{x}{r^2} dy - \frac{y}{r^2} dx\right) = \left(\partial_x \frac{x}{r^2} + \partial_y \frac{y}{r^2}\right) dx \wedge dy = \frac{(r^2 - 2x^2) + (r^2 - 2y^2)}{r^4} dx \wedge dy = 0.$$

Example 3.5 (Leibniz in action)

$d(x dy \wedge dz) = dx \wedge dy \wedge dz$, and for $f = xyz$: $d(f dx) = df \wedge dx = xz dy \wedge dx + xy dz \wedge dx = -xz dx \wedge dy + xy dz \wedge dx$.

Idea Only the dx^i that are not already present survive: d of a form adds the missing directions.

d on a manifold

Definition 3.6 (Exterior derivative on M)

For $\omega \in \Omega^k(M)$ define $d\omega \in \Omega^{k+1}(M)$ chart by chart: $(d\omega)_\alpha := d(\omega_\alpha)$ on $\varphi_\alpha(U_\alpha)$.

Lemma 3.7 (Well defined)

The local forms $d(\omega_\alpha)$ satisfy the transition rule, so $d\omega$ is a well-defined form. Properties (i)–(iv) hold on manifolds, for every smooth $F : M \rightarrow N$.

Proof With $\tau = \varphi_\beta \circ \varphi_\alpha^{-1}$: $d(\omega_\alpha) = d(\tau^* \omega_\beta) = \tau^* d(\omega_\beta)$ by (iv) on $\mathbb{R}^m$. This is the transition rule for the family $d(\omega_\alpha)$. All four properties are local and hold in charts.

Closed, exact, and the de Rham groups

Definition 3.8 (de Rham cohomology)

ω is **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$. Since $d^2 = 0$, exact forms are closed, and

$$H^k(M) = \frac{\ker(d : \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \rightarrow \Omega^k(M))}.$$

Example 3.9 (Degree zero)

$H^0(M)$ is the space of locally constant functions: $\cong \mathbb{R}^c$ if M has c components.

Proof There are no exact 0-forms. $df = 0$ means all partial derivatives vanish in every chart, so f is locally constant; on a connected set it is then constant.

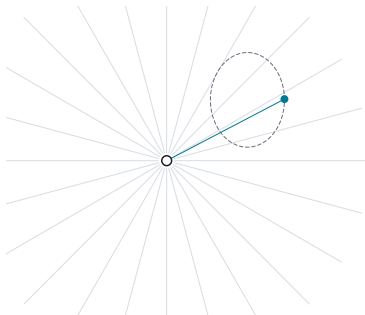
Closed but not exact

Example 3.10 (The angle form)

On $\mathbb{R}^2 \setminus \{0\}$,

$$d\theta := \frac{x dy - y dx}{x^2 + y^2}$$

is closed. It is not exact: its integral over the unit circle is 2π , while an exact form df integrates to 0 over every closed curve (proved in Section 6).



$$d\theta = \frac{x dy - y dx}{x^2 + y^2}$$



$$\int_{\gamma} d\theta = +0.00$$

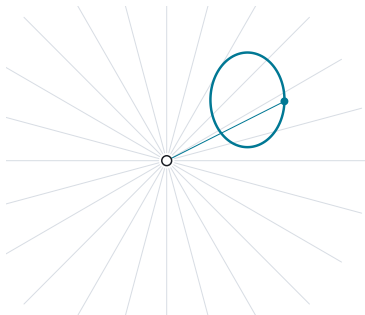
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$$d\theta = \frac{x dy - y dx}{x^2 + y^2}$$



$$\int_{\gamma} d\theta = -0.01$$

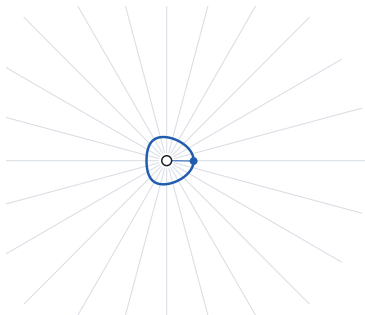
Closed but not exact

Example 3.10 (The angle form)

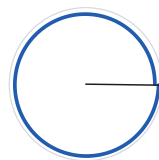
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$$d\theta = \frac{x dy - y dx}{x^2 + y^2}$$



$$\int_{\gamma} d\theta = +6.27$$

4

Smooth bumps and partitions of unity

From local to global

- Integrals, metrics and volume forms are easy to write down **in one chart**.
- To assemble them we cut a global object into pieces, each supported in one chart, that add back up to the whole object.
- The cutting functions must be smooth, so that the pieces are smooth forms again. Analytic functions cannot do this. Smooth functions can, because there are smooth functions that vanish on a whole half-line without vanishing identically.

A function that is flat at 0

Lemma 4.1 (A function that is flat at 0)

$f(t) = e^{-1/t}$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$ defines a smooth function on $\mathbb{R}$.

Proof On $t > 0$, by induction $f^{(n)}(t) = p_n(1/t) e^{-1/t}$ for polynomials p_n ($p_{n+1}(s) = s^2(p_n(s) - p'_n(s))$). Since $s^N e^{-s} \rightarrow 0$ as $s \rightarrow \infty$, every $f^{(n)}(t) \rightarrow 0$ as $t \downarrow 0$, and by induction each $f^{(n)}$ is differentiable at 0 with $f^{(n+1)}(0) = \lim_{t \downarrow 0} f^{(n)}(t)/t = 0$.

Corollary 4.2 (Smooth step, smooth bump)

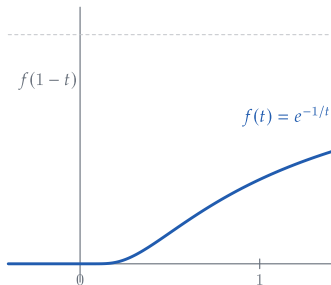
$g(t) = \frac{f(t)}{f(t) + f(1-t)}$ is smooth, 0 on $t \leq 0$ and 1 on $t \geq 1$. For $0 < r < R$ the function

$$\chi(x) = g\left(\frac{R - |x|}{R - r}\right)$$

is smooth on $\mathbb{R}^m$, equal to 1 on B_r and to 0 outside B_R .

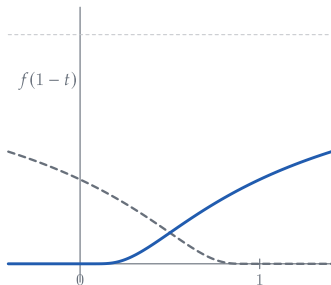
Proof $f(t)$ and $f(1-t)$ never vanish together, so the denominator is positive. χ is smooth where $x \mapsto |x|$ is, that is away from 0; on B_r it is constant 1.

Building a bump



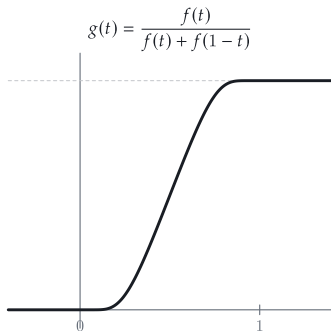
$e^{-1/t}$ leaves 0 with every derivative zero; divided by itself plus its mirror image it becomes a step, and turned about the axis a bump.

Building a bump



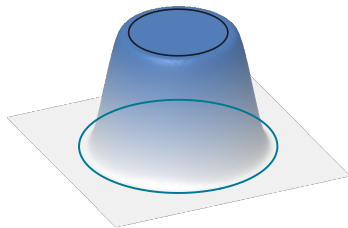
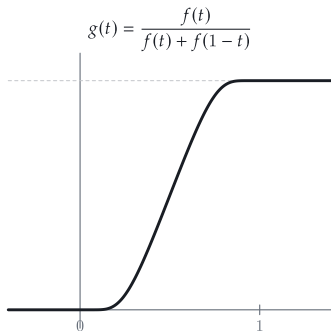
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Building a bump



$e^{-1/t}$ leaves 0 with every derivative zero; divided by itself plus its mirror image it becomes a step, and turned about the axis a bump.

Building a bump



$$\chi = 1 \text{ on } B_r, \text{ supp } \chi \subset \overline{B_R}$$

$e^{-1/t}$ leaves 0 with every derivative zero; divided by itself plus its mirror image it becomes a step, and turned about the axis a bump.

Partitions of unity

Definition 4.3 (Partition of unity)

A **partition of unity** on M is a family $(\rho_\alpha)_{\alpha \in A}$ of smooth functions $\rho_\alpha : M \rightarrow [0, 1]$ such that

- (i) the supports are **locally finite**: every point has a neighbourhood meeting only finitely many $\text{supp } \rho_\alpha$;
- (ii) $\sum_\alpha \rho_\alpha \equiv 1$ (a finite sum near every point).

It is **subordinate** to an open cover $(U_\alpha)_{\alpha \in A}$ if $\text{supp } \rho_\alpha \subset U_\alpha$ for every α .

Theorem 4.4 (Partitions of unity exist)

Let M be a smooth manifold (Hausdorff, second countable). Every open cover $(U_\alpha)_{\alpha \in A}$ of M has a partition of unity subordinate to it, indexed by the same set A .

Proof, steps 1 and 2

1

Exhaustion by compact sets. There are compact $K_1 \subset \text{int } K_2 \subset K_2 \subset \text{int } K_3 \subset \dots$ with $\bigcup_j K_j = M$. Take a countable basis $B_1, B_2, \dots$ of open sets with compact closure (second countable and locally Euclidean). Let $K_1 = \overline{B_1}$, and let K_{j+1} be the closure of the union of finitely many basis sets that cover the compact set $K_j \cup \overline{B_{j+1}}$.



2

Finitely many bumps per shell. Put $K_0 = K_{-1} = \emptyset$. The shell $A_j = K_j \setminus \text{int } K_{j-1}$ is compact and lies in the open set $W_j = \text{int } K_{j+1} \setminus K_{j-2}$. For $p \in A_j$ choose α with $p \in U_\alpha$, a chart ball $B \ni p$ with $\overline{B} \subset W_j \cap U_\alpha$, and a bump $\chi_B \geq 0$ supported in B and positive on a smaller ball $B' \ni p$. Finitely many B' cover A_j .



four caps

Proof, steps 3 and 4

3

Normalise. All shells together give countably many bumps χ_i , with $\text{supp } \chi_i \subset U_{\alpha(i)}$. The neighbourhood $\text{int } K_n$ of a point meets W_j only for $j \leq n+1$, so the supports are locally finite. Hence $\kappa = \sum_i \chi_i$ is smooth, and $\kappa > 0$ because the balls B' cover every shell. Put $\psi_i = \chi_i/\kappa$: then $\sum_i \psi_i = 1$.

4

Regroup by the cover. Let $\rho_\alpha = \sum_{i: \alpha(i)=\alpha} \psi_i$ (zero if there is no such i). Locally only finitely many terms are nonzero, so ρ_α is smooth. A locally finite union of closed sets is closed, so $\text{supp } \rho_\alpha = \bigcup_{\alpha(i)=\alpha} \text{supp } \psi_i \subset U_\alpha$. These supports are locally finite, and $\sum_\alpha \rho_\alpha = \sum_i \psi_i = 1$.



four caps

A partition of unity, seen



$$\sum_{\alpha} \rho_{\alpha} \equiv 1$$

Raw bumps on four arcs overlap and do not add up to 1; divided by their sum they stack into a ring of constant thickness. On S^2 : four caps, coloured by $\sum_{\alpha} \rho_{\alpha} c_{\alpha}$.

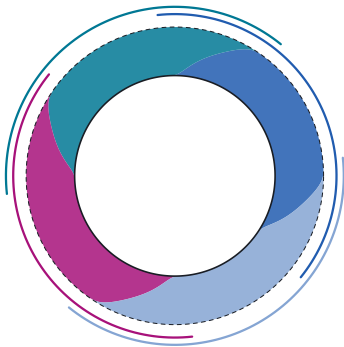
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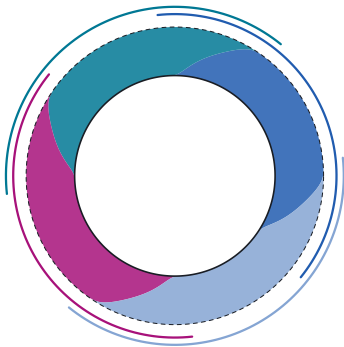
A partition of unity, seen



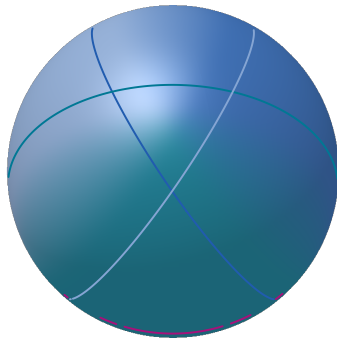
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A partition of unity, seen



$$\sum_{\alpha} \rho_{\alpha} \equiv 1$$



$$S^2 = U_1 \cup \dots \cup U_4$$

Raw bumps on four arcs overlap and do not add up to 1; divided by their sum they stack into a ring of constant thickness. On S^2 : four caps, coloured by $\sum_{\alpha} \rho_{\alpha} c_{\alpha}$.

A first use: smooth Urysohn

Corollary 4.5 (Smooth Urysohn)

For disjoint closed sets $A, B \subset M$ there is a smooth $f : M \rightarrow [0, 1]$ with $f|_A = 1$ and $f|_B = 0$.

Proof $\{M \setminus B, M \setminus A\}$ is an open cover. Take a subordinate partition (ρ_1, ρ_2) and $f = \rho_1$: on B , $\rho_1 = 0$ because $\text{supp } \rho_1 \subset M \setminus B$; on A , $\rho_2 = 0$, so $\rho_1 = 1$.

Idea Cut-off functions like this let us extend a form defined near a compact set to all of M , and localise integrals.

A second use: Riemannian metrics

Corollary 4.6 (Riemannian metrics exist)

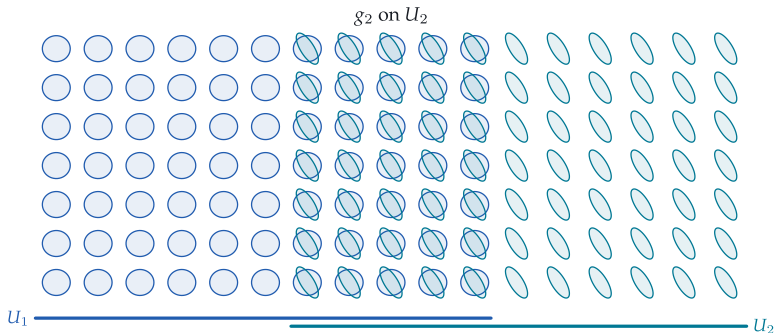
Every manifold carries a Riemannian metric.

Proof For an atlas $(U_\alpha, \varphi_\alpha)$ and a subordinate partition (ρ_α) let

$$g = \sum_{\alpha} \rho_{\alpha} \varphi_{\alpha}^* \langle \cdot, \cdot \rangle, \quad g_p(v, w) = \sum_{\alpha: p \in U_{\alpha}} \rho_{\alpha}(p) \langle d\varphi_{\alpha}(p)v, d\varphi_{\alpha}(p)w \rangle.$$

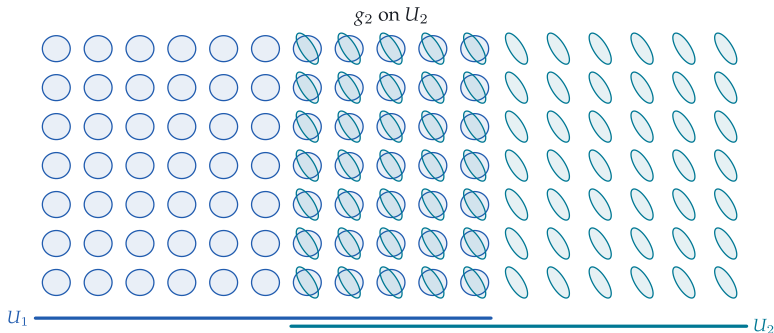
It is smooth (a locally finite sum) and symmetric. For $v \neq 0$ every term of $g_p(v, v)$ is ≥ 0 , and the terms with $\rho_{\alpha}(p) > 0$ are > 0 because $d\varphi_{\alpha}(p)$ is injective; so g_p is positive definite.

Patching metrics



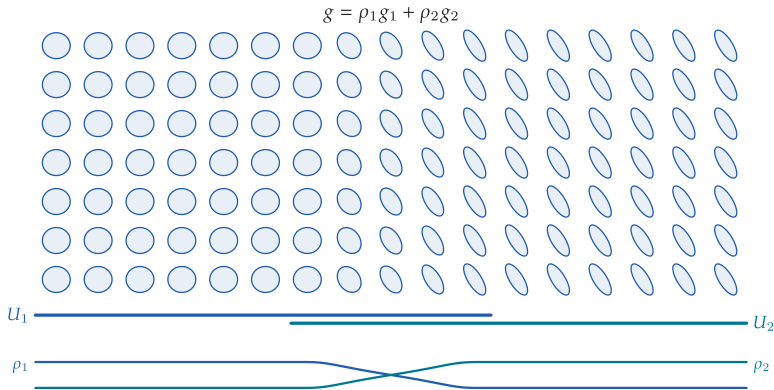
Each metric is drawn as a field of unit ellipses $\{v : g(v, v) = r^2\}$. Across the overlap, $\rho_1 g_1 + \rho_2 g_2$ interpolates between them, and every blend is again an ellipse.

Patching metrics



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5

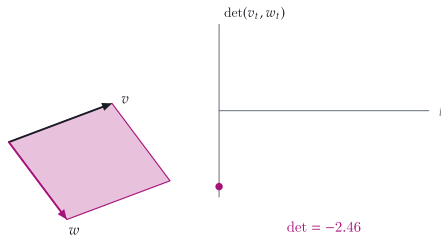
Orientation

Orienting a vector space

Definition 5.1 (Orientation)

Two ordered bases of V are **equally oriented** if the matrix taking one to the other has positive determinant. An **orientation** is one of the two equivalence classes.

Equivalently: a choice of one of the two components of $\Lambda^m V^* \setminus \{0\}$; a basis is positive if $\omega(v_1, \dots, v_m) > 0$ for ω in that component.



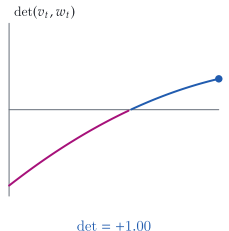
A positive basis deforms to (e_1, e_2) through bases. A negative one cannot: $\det$ has to pass through 0.

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A positive basis deforms to (e_1, e_2) through bases. A negative one cannot: $\det$ has to pass through 0.

Orienting a manifold

Definition 5.2 (Oriented atlas, orientation)

An atlas $(U_\alpha, \varphi_\alpha)$ is **oriented** if all transition maps have positive Jacobian determinant,

$$\det D(\varphi_\beta \circ \varphi_\alpha^{-1})(x) > 0 \quad \text{for all } \alpha, \beta, x \in \varphi_\alpha(U_\alpha \cap U_\beta).$$

M is **orientable** if it has an oriented atlas. An **orientation** is a maximal oriented atlas; equivalently, an orientation of every $T_p M$ that is given locally by the coordinate bases $(\partial_1, \dots, \partial_m)$ of charts.

Example 5.3 (The sphere)

$\{\varphi_N, \rho \circ \varphi_S\}$ with the reflection $\rho(y^1, y^2) = (y^1, -y^2)$ is an oriented atlas of S^2 : the transition has determinant $+|x|^{-4}$.

Orientable means: there is a volume form

Lemma 5.4 (Volume forms)

A nowhere vanishing m -form on M^m is called a **volume form**. M is orientable if and only if it has a volume form.

Proof ($\Leftarrow$) Let ω be a volume form and take an atlas of connected charts. In a chart $\omega_\alpha = f_\alpha dx^1 \wedge \cdots \wedge dx^m$ with $f_\alpha \neq 0$, so f_α has one sign. Where it is negative, replace φ_α by its composition with $(x^1, \dots, x^m) \mapsto (-x^1, x^2, \dots, x^m)$; now all $f_\alpha > 0$. The transition rule gives

$$f_\alpha = (f_\beta \circ \tau) \det D\tau, \quad \tau = \varphi_\beta \circ \varphi_\alpha^{-1},$$

so $\det D\tau = f_\alpha / (f_\beta \circ \tau) > 0$: the atlas is oriented.

Orientable means: there is a volume form, continued

Proof ($\Rightarrow$) Take an oriented atlas $(U_\alpha, \varphi_\alpha)$ and a subordinate partition of unity (ρ_α) , and set

$$\omega = \sum_{\alpha} \rho_{\alpha} \varphi_{\alpha}^* (dx^1 \wedge \cdots \wedge dx^m),$$

each term extended by 0 outside U_α (smooth, since $\text{supp } \rho_\alpha \subset U_\alpha$). Fix p and a basis $(v_1, \dots, v_m)$ of $T_p M$ that is positive for one chart around p , hence for all of them. Then

$$\omega_p(v_1, \dots, v_m) = \sum_{\alpha: p \in U_\alpha} \rho_\alpha(p) \det [d\varphi_\alpha(p)v_1 \cdots d\varphi_\alpha(p)v_m].$$

Every term is ≥ 0 , and some $\rho_\alpha(p) > 0$; so $\omega_p \neq 0$.

Idea Positivity is preserved by sums with non-negative weights. This is the first place where the partition of unity does real work.

Examples and a non-example

Example 5.5 (Spheres)

On $S^m \subset \mathbb{R}^{m+1}$, $\omega_x(v_1, \dots, v_m) = \det[x \ v_1 \cdots v_m]$ is a volume form:

$$\omega = \sum_{i=1}^{m+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{m+1} \Big|_{S^m}.$$

It does not vanish, because for a basis $v_1, \dots, v_m$ of $T_x S^m = x^\perp$ the vectors $x, v_1, \dots, v_m$ are a basis of $\mathbb{R}^{m+1}$.

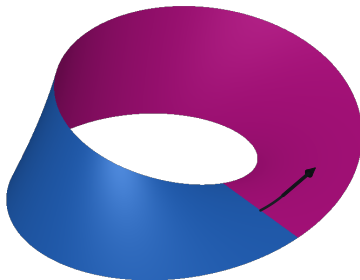
A non-example

Example 5.6 (The Möbius band is not orientable)

Suppose ω were a volume form. Carry a frame $(e_1(t), e_2(t))$ continuously once along the core circle. It returns as $(e_1, -e_2)$, so $t \mapsto \omega(e_1(t), e_2(t))$ is continuous, never 0, and changes sign between $t = 0$ and $t = 1$, which contradicts the intermediate value theorem.

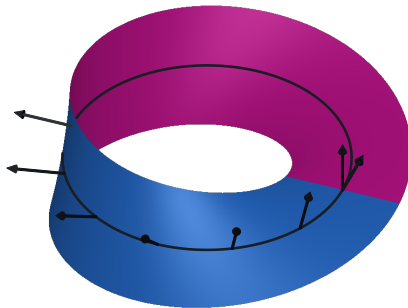
Idea Locally every manifold is orientable (a chart is). Orientability is a global property: it can fail only along a closed loop, never in a single chart.

One side only



Front and back of the band in two colours. They meet along a line: after one turn the frame comes back with its normal reversed.

One side only



$n \mapsto -n$ after one turn

Front and back of the band in two colours. They meet along a line: after one turn the frame comes back with its normal reversed.

The orientation of the boundary

Definition 5.7 (Outward normal first)

Model half-space: $\mathbb{H}^m = \{x \in \mathbb{R}^m : x^1 \leq 0\}$, so that e_1 points out of $\mathbb{H}^m$. For $p \in \partial M$ let n be an outward vector. A basis $(v_2, \dots, v_m)$ of $T_p \partial M$ is **positive** if $(n, v_2, \dots, v_m)$ is positive in $T_p M$.

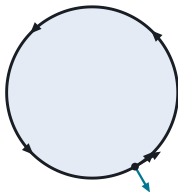
Lemma 5.8 (Boundaries are oriented)

If M is oriented, this rule defines an orientation of ∂M .

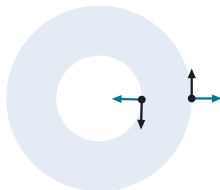
Proof Restrict the boundary charts of an oriented atlas to $\partial M = \{x^1 = 0\}$. A transition τ maps $\mathbb{H}^m$ to $\mathbb{H}^m$ and $\partial \mathbb{H}^m$ to itself, so $\tau^1(0, x') = 0$ and $\partial_1 \tau^1 > 0$ on the boundary, while $\partial_j \tau^1 = 0$ there for $j \geq 2$. Expanding $\det D\tau$ along the first row: $\det D\tau = \partial_1 \tau^1 \cdot \det D(\tau|_{\partial})$, so $\det D(\tau|_{\partial}) > 0$. In these charts e_1 is outward, so the restricted atlas gives exactly the rule above.

Outward normal first

$$(n_{\text{out}}, v_2, \dots, v_m) \text{ positive} \iff (v_2, \dots, v_m) \text{ positive on } \partial M$$



disc

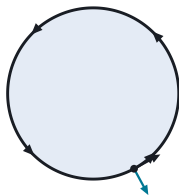


annulus

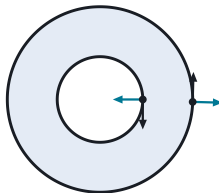
With $\mathbb{H}^m = \{x^1 \leq 0\}$ the boundary $\partial\mathbb{H}^m = \{0\} \times \mathbb{R}^{m-1}$ carries its standard orientation: no sign $(-1)^m$ will appear in Stokes' theorem.

Outward normal first

$$(n_{\text{out}}, v_2, \dots, v_m) \text{ positive} \iff (v_2, \dots, v_m) \text{ positive on } \partial M$$



disc

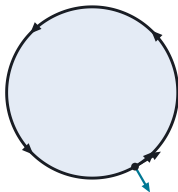


annulus

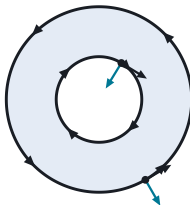
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Outward normal first

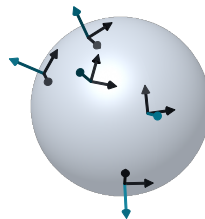
$$(n_{\text{out}}, v_2, \dots, v_m) \text{ positive} \iff (v_2, \dots, v_m) \text{ positive on } \partial M$$



disc



annulus



ball

With $\mathbb{H}^m = \{x^1 \leq 0\}$ the boundary $\partial\mathbb{H}^m = \{0\} \times \mathbb{R}^{m-1}$ carries its standard orientation: no sign $(-1)^m$ will appear in Stokes' theorem.

6 *The integral*

Integrating in one chart

Definition 6.1 (Integral over an open set)

Let $U \subset \mathbb{R}^m$ or $U \subset \mathbb{H}^m$ be open and $\omega = f dx^1 \wedge \cdots \wedge dx^m$ with f compactly supported in U . Then $\int_U \omega := \int_U f dx^1 \cdots dx^m$ (Lebesgue measure).

Lemma 6.2 (Change of variables for forms)

Let $F : U \rightarrow V$ be a diffeomorphism of connected open sets and $\omega \in \Omega_c^m(V)$. Then

$$\int_U F^* \omega = \pm \int_V \omega,$$

with $+$ if F preserves orientation ($\det DF > 0$) and $-$ if it reverses it.

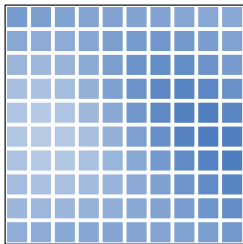
Change of variables: proof

Proof $F^*\omega = (f \circ F) \det DF \, dx$. The classical formula is $\int_V f \, dy = \int_U (f \circ F) |\det DF| \, dx$, and $\det DF$ never vanishes, so it has constant sign on the connected set U .

Idea Forms see $\det DF$, the classical integral sees $|\det DF|$. As long as all transition maps have positive determinant the two agree; that is what an orientation will provide.

Change of variables

U

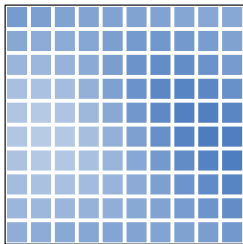


$$\int_U F^* \omega \approx +2.291$$

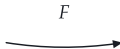
Each cell of U carries $g(F(x)) \det DF | \text{cell} |$; carried to V the cells add up to $\int_V \omega$. A reflection turns every cell purple and the sum changes sign.

Change of variables

U



$V = F(U)$



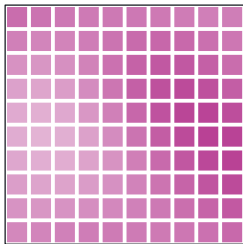
$$\int_U F^* \omega \approx +2.291$$

$$\int_V \omega \approx +2.291$$

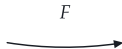
Each cell of U carries $g(F(x)) \det DF | \text{cell} |$; carried to V the cells add up to $\int_V \omega$. A reflection turns every cell purple and the sum changes sign.

Change of variables

U



$V = F(U)$



$$\int_U F^* \omega \approx -2.291$$

$$\int_V \omega \approx +2.291$$

Each cell of U carries $g(F(x)) \det DF | \text{cell} |$; carried to V the cells add up to $\int_V \omega$. A reflection turns every cell purple and the sum changes sign.

The integral on an oriented manifold

Definition 6.3 ($\int_M \omega$)

Let M be oriented, $(U_\alpha, \varphi_\alpha)$ an oriented atlas, (ρ_α) a subordinate partition of unity and $\omega \in \Omega_c^m(M)$ compactly supported. Define

$$\int_M \omega := \sum_{\alpha} \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^* (\rho_\alpha \omega).$$

- Each term is an integral of a compactly supported form on an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$.
- The sum is finite: the compact set $\text{supp } \omega$ meets only finitely many of the locally finite supports.
- It remains to show that the result depends neither on the atlas nor on the partition.

The integral is well defined

Lemma 6.4 (Independence of the choices)

Another oriented atlas (V_β, ψ_β) of the same orientation, with a subordinate partition (θ_β) , gives the same value.

The idea of the proof Refine both cuts at once: the pieces $\rho_\alpha \theta_\beta \omega$ each live in one chart of either atlas. Computed in φ_α or in ψ_β , their integrals agree by the change of variables lemma, because the transition is orientation preserving. Summing over β first gives the first definition, over α first the second.

The integral is well defined: proof

1

Insert $1 = \sum_{\beta} \theta_{\beta}$ (a finite sum on $\text{supp } \omega$):

$$\int_{\varphi_{\alpha}(U_{\alpha})} (\varphi_{\alpha}^{-1})^* (\rho_{\alpha} \omega) = \sum_{\beta} \int_{\varphi_{\alpha}(U_{\alpha} \cap V_{\beta})} (\varphi_{\alpha}^{-1})^* (\rho_{\alpha} \theta_{\beta} \omega).$$



2

The form $\rho_{\alpha} \theta_{\beta} \omega$ lives in $U_{\alpha} \cap V_{\beta}$. The map $\psi_{\beta} \circ \varphi_{\alpha}^{-1}$ preserves orientation, so by the change of variables lemma

$$\int_{\varphi_{\alpha}(\dots)} (\varphi_{\alpha}^{-1})^* (\rho_{\alpha} \theta_{\beta} \omega) = \int_{\psi_{\beta}(\dots)} (\psi_{\beta}^{-1})^* (\rho_{\alpha} \theta_{\beta} \omega).$$



3

Sum over α and β . Now use $\sum_{\alpha} \rho_{\alpha} = 1$:

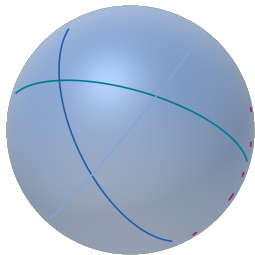
$$\sum_{\alpha} \int_{\varphi_{\alpha}} (\rho_{\alpha} \omega) = \sum_{\alpha, \beta} \int (\rho_{\alpha} \theta_{\beta} \omega) = \sum_{\beta} \int_{\psi_{\beta}} (\theta_{\beta} \omega).$$

The double sum is symmetric in the two choices.

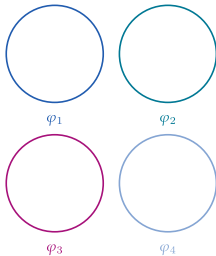


two covers, one total

Gluing chart integrals



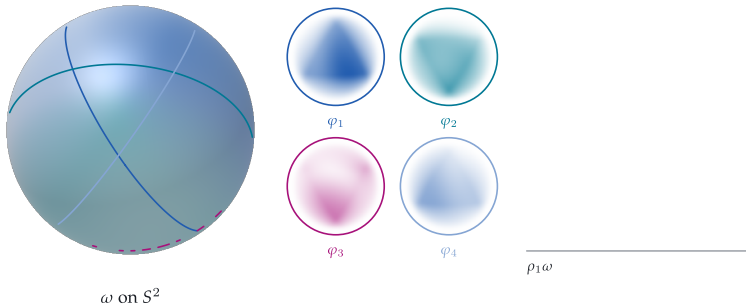
ω on S^2



$$\int_{S^2} \omega = 12.566$$

The partition of unity cuts ω into four pieces; each is integrated in its own chart. A second, turned cover cuts different pieces, and they add up to the same total.

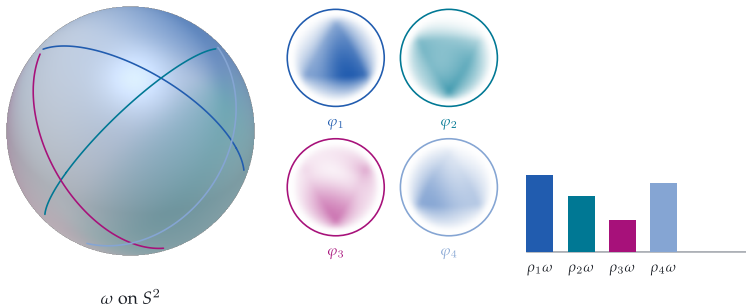
Gluing chart integrals



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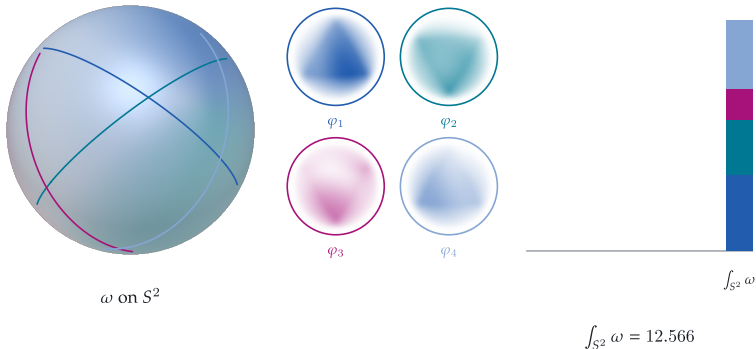
Gluing chart integrals



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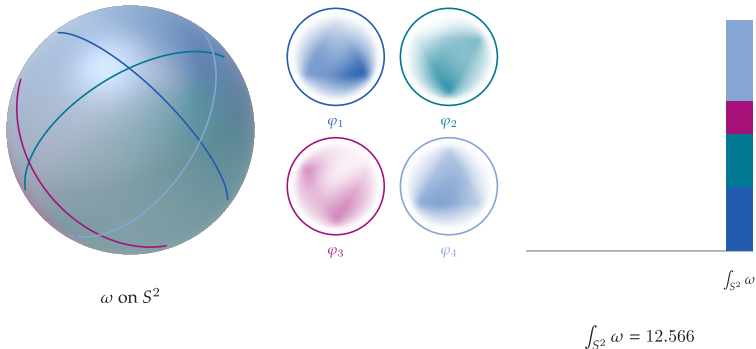
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Gluing chart integrals



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Gluing chart integrals



The partition of unity cuts ω into four pieces; each is integrated in its own chart. A second, turned cover cuts different pieces, and they add up to the same total.

Properties of the integral

Proposition 6.5 (Rules)

For $\omega, \eta \in \Omega_c^m(M)$ on an oriented M :

- (i) $\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta$;
- (ii) reversing the orientation of M changes the sign of $\int_M \omega$;
- (iii) for an orientation preserving diffeomorphism $F : M \rightarrow N$, $\int_M F^* \omega = \int_N \omega$;
- (iv) if $\text{supp } \omega \subset U_\alpha$, then $\int_M \omega = \int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^* \omega$.

Proof (i) term by term. (ii): composing each chart with a reflection reverses it, and every term changes sign by the change of variables lemma. (iii): $(F^{-1}(U_\alpha), \varphi_\alpha \circ F)$ is an oriented atlas of M with partition $(\rho_\alpha \circ F)$, and its terms for $F^* \omega$ are those for ω . (iv): use any partition with $\rho_\alpha = 1$ on $\text{supp } \omega$, which is allowed by independence.

Integrating over submanifolds and curves

Definition 6.6 (Lower dimensions)

For an oriented compact k -dimensional submanifold $S \subset M$ with inclusion ι and $\omega \in \Omega^k(M)$:
 $\int_S \omega := \int_S \iota^* \omega$. For a smooth curve $\gamma : [a, b] \rightarrow M$ and a 1-form α : $\int_\gamma \alpha := \int_a^b \alpha_{\gamma(t)}(\dot{\gamma}(t)) dt$.

Lemma 6.7 (Reparametrisation)

If $h : [c, d] \rightarrow [a, b]$ is a diffeomorphism, then $\int_{\gamma \circ h} \alpha = \pm \int_\gamma \alpha$, with $+$ if h is increasing.

Proof $\alpha_{\gamma(h(s))}(\frac{d}{ds} \gamma(h(s))) = \alpha_{\gamma(h(s))}(\dot{\gamma}(h(s))) h'(s)$; substitute $t = h(s)$.

The angle form, finished

Example 6.8 (The angle form, finished)

For $\gamma(t) = (\cos t, \sin t)$: $d\theta(\dot{\gamma}) = \cos^2 t + \sin^2 t = 1$, so $\int_{\gamma} d\theta = 2\pi$. For exact df : $\int_{\gamma} df = \int_a^b (f \circ \gamma)' dt = f(\gamma(b)) - f(\gamma(a)) = 0$ on closed curves. So $d\theta$ is not exact.

Computing with one parametrisation

Lemma 6.9 (Up to a null set)

Let $W \subset \mathbb{R}^m$ be open and $\phi : W \rightarrow M$ an orientation preserving parametrisation (a diffeomorphism onto an open set) such that $M \setminus \phi(W)$ is closed and has measure zero. Then $\int_M \omega = \int_W \phi^* \omega$.

Proof Let $Z = M \setminus \phi(W)$. Choose smooth $\chi_n : M \rightarrow [0, 1]$ vanishing near Z with $\chi_n \rightarrow 1$ pointwise off Z (smooth Urysohn for $\{d(\cdot, Z) \geq 1/n\}$ and $\{d(\cdot, Z) \leq 1/2n\}$, d a metric for the topology). Then $\int_M \chi_n \omega = \int_W \phi^*(\chi_n \omega)$ by (iii),(iv). In charts, both sides converge by dominated convergence, to $\int_M \omega$ (as Z is a null set) and to $\int_W \phi^* \omega$.

The area of the sphere

Example 6.10 (The sphere)

With spherical coordinates on $(0, \pi) \times (0, 2\pi)$ and σ from Section 2: $\int_{S^2} \sigma = \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi = 4\pi$.

Where we stand

Forms

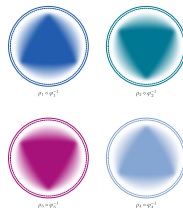
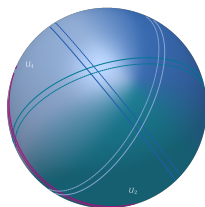
transform with det of the transition, sign included.

Partitions of unity

cut ω into chart pieces that add up to ω .

Orientation

makes every det positive, so $\det = |\det|$.



$$\int_M \omega = \sum_i \int_{\rho_i(U_i)} (\rho_i^{-1})^* (\rho_i^* \omega)$$

7

Stokes' theorem

Stokes' theorem

Theorem 7.1 (Stokes)

Let M be an oriented m -manifold with boundary, ∂M oriented by the outward normal first, and $\omega \in \Omega_c^{m-1}(M)$. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

If $\partial M = \emptyset$, the right-hand side is 0.

Stokes' theorem

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If $\partial M = \emptyset$, the right-hand side is 0.

Known

$m = 1$, $M = [a, b]$: $\int_a^b f' dt = f(b) - f(a)$,
the fundamental theorem of calculus.

Stokes' theorem

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Known

$m = 1$, $M = [a, b]$: $\int_a^b f' dt = f(b) - f(a)$,
the fundamental theorem of calculus.

The trick

On the half-space each term of $d\omega$ is a one-variable derivative: use the fundamental theorem in that variable. Then glue with $\sum_\alpha d\rho_\alpha = d(1) = 0$.

The hypotheses matter

Example 7.2 (Compact support)

On $M = (0, 1)$ (no boundary points) let $\omega = x$, a 0-form without compact support. Then $\int_M d\omega = \int_0^1 dx = 1$, while $\partial M = \emptyset$. Stokes fails: mass escapes through the missing end points.

Example 7.3 (Orientation)

On the Möbius band there is no integral of 2-forms at all: an oriented atlas is needed to make the chart integrals agree. (One can integrate *densities* $|\omega|$ there, but d does not act on densities.)

Example 7.4 (Signs)

For $D = \{|x| \leq 1\} \subset \mathbb{R}^2$ and $\omega = x dy$: $\int_D d\omega = \pi$, and with the boundary run counter-clockwise, $\int_{\partial D} x dy = \int_0^{2\pi} \cos^2 t dt = \pi$. Running it clockwise would give $-\pi$.

Step 1: the half-space

Let $M = \mathbb{H}^m = \{x^1 \leq 0\}$ and write

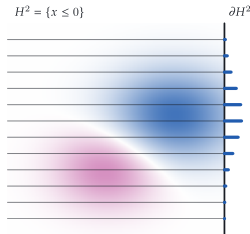
$$\omega = \sum_{i=1}^m f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m,$$

f_i compactly supported. Then

$$d\omega = \sum_{i=1}^m (-1)^{i-1} \partial_i f_i dx^1 \wedge \cdots \wedge dx^m.$$

By Fubini, integrate $\partial_i f_i$ first in x^i :

- $i \geq 2$: $\int_{\mathbb{R}} \partial_i f_i dx^i = 0$ (compact support);
- $i = 1$: $\int_{-\infty}^0 \partial_1 f_1 dx^1 = f_1(0, x')$.



Step 1: the half-space

Let $M = \mathbb{H}^m = \{x^1 \leq 0\}$ and write

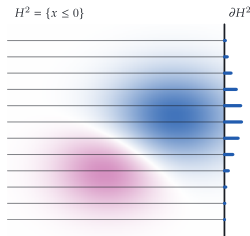
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- $i \geq 2$: $\int_{\mathbb{R}} \partial_i f_i dx^i = 0$ (compact support);
- $i = 1$: $\int_{-\infty}^0 \partial_1 f_1 dx^1 = f_1(0, x')$.



$$\int_{H^2} d\omega = \int_{\partial H^2} f(0, y) dy$$

Step 1, finished

- Hence $\int_{\mathbb{H}^m} d\omega = \int_{\mathbb{R}^{m-1}} f_1(0, x^2, \dots, x^m) dx^2 \cdots dx^m$.
- On $\partial\mathbb{H}^m = \{x^1 = 0\}$ we have $\iota^* dx^1 = 0$, so only the $i = 1$ term survives:
 $\iota^* \omega = f_1(0, x') dx^2 \wedge \cdots \wedge dx^m$.
- e_1 is the outward normal, and $(e_1, e_2, \dots, e_m)$ is positive. So $(e_2, \dots, e_m)$ is the boundary orientation, which is the standard orientation of $\mathbb{R}^{m-1}$, and

$$\int_{\partial\mathbb{H}^m} \omega = \int_{\mathbb{R}^{m-1}} f_1(0, x') dx' = \int_{\mathbb{H}^m} d\omega.$$

- Without boundary ($M = \mathbb{R}^m$) every term vanishes: $\int_{\mathbb{R}^m} d\omega = 0$.

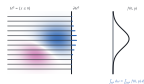
Step 2: forms supported in one chart

2

One chart. Suppose $\text{supp } \omega \subset U$ for an oriented chart $\varphi : U \rightarrow \varphi(U) \subset \mathbb{H}^m$ (or $\mathbb{R}^m$). By the definition of the integral and naturality of d ,

$$\int_M d\omega = \int_{\varphi(U)} (\varphi^{-1})^* d\omega = \int_{\mathbb{H}^m} d((\varphi^{-1})^* \omega),$$

where the compactly supported form $(\varphi^{-1})^* \omega$ is extended by 0. Step 1 turns this into $\int_{\partial \mathbb{H}^m} (\varphi^{-1})^* \omega$.

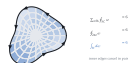


the half-space

2'

The boundary. The chart restricts to an oriented chart of ∂M for the outward-normal-first orientation (Section 5), so $\int_{\partial \mathbb{H}^m} (\varphi^{-1})^* \omega = \int_{\partial M} \omega$. For an interior chart ($\varphi(U) \subset \mathbb{R}^m$) both sides are 0.

Step 3: gluing with a partition of unity



inner edges cancel

3

Cut. Take an oriented atlas and a subordinate partition of unity. Only finitely many ρ_α meet the compact set $\text{supp } \omega$, so $\omega = \sum_\alpha \rho_\alpha \omega$ is a finite sum of forms supported in single charts.



4

The derivatives add up. By the Leibniz rule

$$\sum_\alpha d(\rho_\alpha \omega) = (\sum_\alpha d\rho_\alpha) \wedge \omega + \sum_\alpha \rho_\alpha d\omega = d\left(\sum_\alpha \rho_\alpha\right) \wedge \omega + d\omega = d\omega,$$

since $\sum_\alpha \rho_\alpha \equiv 1$ has derivative 0.

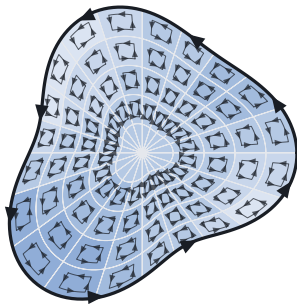


✓

Add. Apply Step 2 to each piece:

$$\int_M d\omega = \sum_\alpha \int_M d(\rho_\alpha \omega) = \sum_\alpha \int_{\partial M} \rho_\alpha \omega = \int_{\partial M} \omega.$$

Inner edges cancel



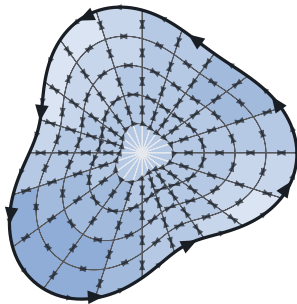
$$\sum_{\text{cells}} \oint_{\partial C} \omega = 6.41$$

$$\oint_{\partial M} \omega = 6.41$$

$$\int_M d\omega = 6.41$$

Each cell carries its own circulation. Along a shared edge two neighbours run in opposite directions, so the sum over the cells is the circulation around ∂M , and in the limit $\int_M d\omega$.

Inner edges cancel



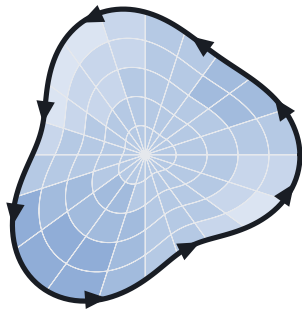
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$$\sum_{\text{cells}} \oint_{\partial C} \omega = 6.41$$

$$\oint_{\partial M} \omega = 6.41$$

$$\int_M d\omega = 6.41$$

inner edges cancel in pairs

Each cell carries its own circulation. Along a shared edge two neighbours run in opposite directions, so the sum over the cells is the circulation around ∂M , and in the limit $\int_M d\omega$.

First consequences

Corollary 7.5 (Closed manifolds)

If M is compact, oriented and $\partial M = \emptyset$, then $\int_M d\eta = 0$ for every $\eta \in \Omega^{m-1}(M)$. In particular a volume form ω compatible with the orientation is never exact, so $H^m(M) \neq 0$.

Proof Stokes with $\partial M = \emptyset$. A compatible volume form has $\omega_\alpha = f_\alpha dx$ with $f_\alpha > 0$ in an oriented atlas, so every term of $\int_M \omega$ is > 0 for some α and ≥ 0 for all.

Example 7.6 (Integrals of closed forms over boundaries)

If $d\omega = 0$ on a compact oriented W with $\partial W = S_1 \sqcup \overline{S_0}$, then $\int_{S_1} \omega = \int_{S_0} \omega$. For curves: $\int_{\gamma_1} d\theta = \int_{\gamma_0} d\theta$ whenever γ_0, γ_1 bound an annulus in $\mathbb{R}^2 \setminus \{0\}$.

Green's theorem and the planimeter

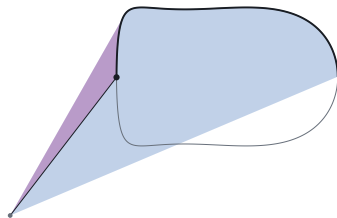
Example 7.7 (Green)

For a compact domain $D \subset \mathbb{R}^2$ with smooth boundary and $\omega = P dx + Q dy$:

$$\oint_{\partial D} P dx + Q dy = \int_D (\partial_x Q - \partial_y P) dx dy.$$

With $\omega = \frac{1}{2}(x dy - y dx)$, $d\omega = dx \wedge dy$:

$$\text{area}(D) = \frac{1}{2} \oint_{\partial D} x dy - y dx.$$



$$\begin{aligned} \frac{1}{2} \oint_{\partial D} x dy - y dx \\ = +4.080 \end{aligned}$$

area of $D = 3.880$

Triangles swept from an outside point: blue while the point turns counter-clockwise about it, purple while it turns back.

Green's theorem and the planimeter

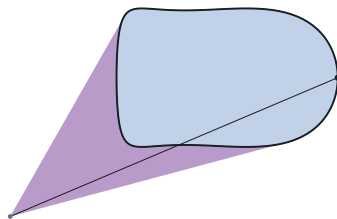
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The divergence theorem

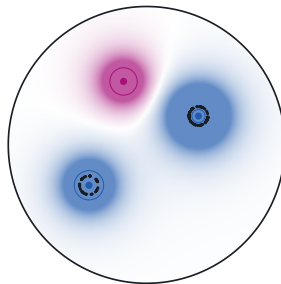
Example 7.8 (Divergence theorem)

For a compact domain $W \subset \mathbb{R}^3$ with smooth boundary and outward unit normal ν :

$$\int_W \operatorname{div} F \, dV = \int_{\partial W} F \cdot \nu \, dA.$$

Take $\omega = \iota_F(dx \wedge dy \wedge dz)$: $d\omega = \operatorname{div} F \, dV$, and on ∂W , $\omega = (F \cdot \nu) \, dA$ because $\iota_F dV(v, w) = \det[F \, v \, w] = F \cdot \nu \, \det[v \, w]$ for v, w tangent.

Example: $F = x$ gives $\int_{S^2} \sigma = 3 \operatorname{vol}(B^3) = 4\pi$.



The same in the plane: the sources' total strength equals the outward flux.

$$\int_D \operatorname{div} F \, dA = 0.000$$

$$\oint_{\partial D} F \cdot \nu \, ds = 0.000$$

blue: sources, purple: sinks

The divergence theorem

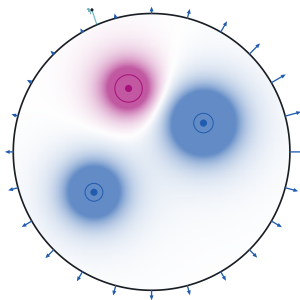
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The same in the plane: the sources' total strength equals the outward flux.

$$\int_D \operatorname{div} F \, dA = 6.790$$

$$\oint_{\partial D} F \cdot \nu \, ds = 6.790$$

blue: sources, purple: sinks

Example 7.9 (Circulation and curl)

For an oriented compact surface $\Sigma \subset \mathbb{R}^3$ with boundary and a vector field F :

$$\oint_{\partial\Sigma} F \cdot d\mathbf{r} = \int_{\Sigma} (\operatorname{curl} F) \cdot \nu \, dA.$$

Here $\omega = F_1 dx + F_2 dy + F_3 dz$, $d\omega = \operatorname{curl} F \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy)$, and on Σ the 2-form $G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$ restricts to $(G \cdot \nu) dA$.

Idea Fundamental theorem of calculus, Green, Gauss and Kelvin–Stokes are one statement:

$$\int_M d\omega = \int_{\partial M} \omega.$$

No smooth retraction of the ball onto its boundary

Theorem 7.10 (No retraction)

There is no smooth map $r : B^m \rightarrow S^{m-1}$ with $r|_{S^{m-1}} = \text{id}$.

Proof Let σ be the volume form of S^{m-1} from Section 5, so $\int_{S^{m-1}} \sigma > 0$. If r existed,

$$0 < \int_{S^{m-1}} \sigma = \int_{\partial B^m} r^* \sigma = \int_{B^m} d(r^* \sigma) = \int_{B^m} r^*(d\sigma) = 0,$$

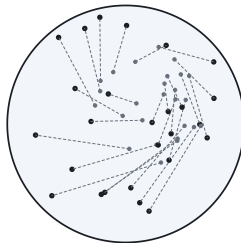
because $d\sigma$ is an m -form on the $(m-1)$ -manifold S^{m-1} , hence 0. The first equality uses $r|_{\partial B} = \text{id}$, the second Stokes, the third naturality of d .

Brouwer's fixed point theorem

Corollary 7.11 (Brouwer, smooth version)

Every smooth $f : B^m \rightarrow B^m$ has a fixed point.

Idea If f moved every point, the ray from $f(x)$ through x would push the ball onto its boundary circle. Such a retraction does not exist.



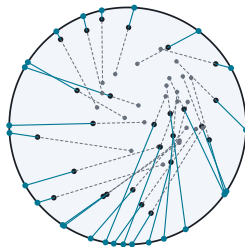
points x (black), $f(x)$ (grey)
the ray from $f(x)$ through x
leaves the disc at $r(x) \in S^1$

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$r|_{S^1} = \text{id}$; smooth if $f(x) \neq x$

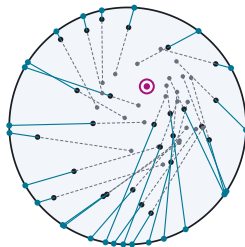
$$0 < \int_{S^1} \sigma = \int_{S^1} r^* \sigma = \int_B r^* d\sigma = 0$$

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a fixed point: no ray

Brouwer's fixed point theorem: proof

Proof Suppose $f(x) \neq x$ for all x . Put $u(x) = \frac{x - f(x)}{|x - f(x)|}$ and

$$r(x) = x + t(x) u(x), \quad t = -x \cdot u + \sqrt{D}, \quad D = (x \cdot u)^2 + 1 - |x|^2,$$

the point where the ray from $f(x)$ through x leaves the ball ($t \geq 0$ solves $|x + tu| = 1$).

- $D > 0$ everywhere: for $|x| < 1$ clearly; for $|x| = 1$, $x \cdot u = \frac{1 - x \cdot f(x)}{|x - f(x)|} > 0$, because $x \cdot f(x) \leq |f(x)| \leq 1$ with equality only if $f(x) = x$.
- Hence $\sqrt{D}$, t and r are smooth, r maps into S^{m-1} , and for $|x| = 1$: $\sqrt{D} = x \cdot u$, so $t = 0$ and $r(x) = x$.

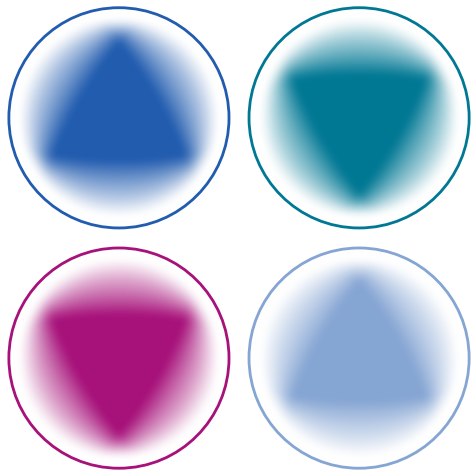
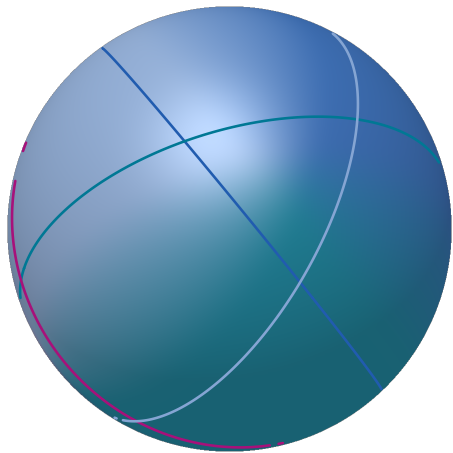
So r is a smooth retraction of B^m onto S^{m-1} , which the previous theorem rules out.

What we built

1. Alternating forms: $\Phi^* \omega = \det(\Phi) \omega$ in top degree.
2. Differential forms, the pullback and the transition rule.
3. The exterior derivative: natural, $d^2 = 0$; closed versus exact.
4. Smooth bumps and partitions of unity, which exist on every manifold.
5. Orientation: orientable means there is a volume form.
6. The integral $\int_M \omega$, independent of all choices.
7. Stokes: $\int_M d\omega = \int_{\partial M} \omega$, with Green, Gauss, Kelvin–Stokes and Brouwer.

Remark 8.1 (Continuous maps). Brouwer's theorem holds for continuous $f : B^m \rightarrow B^m$: approximate f uniformly by smooth maps into the ball; their fixed points accumulate at a fixed point of f . (Stated without proof.)

Next lecture Which closed forms are exact? Is a volume form determined by its total volume? How much of M does $H^*(M)$ see? Lie derivatives, Moser's trick, Mayer–Vietoris and Poincaré duality.



Thank you.